QUANTUM MEASUREMENTS AND MAPS PRESERVING STRICT CONVEX COMBINATIONS AND PURE STATES

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ABSTRACT. Let $\mathcal{S}(H)$ be the convex set of all states (i.e., the positive operators with trace one) on a complex Hilbert space H. It is shown that a map $\psi : \mathcal{S}(H) \to \mathcal{S}(K)$ with $2 \leq \dim H < \infty$ preserves pure states and strict convex combinations (i.e., for any ρ_1, ρ_2 and $0 < t < 1$, there exists $0 < s < 1$ such that $\psi(t\rho_1 + (1-t)\rho_2) = s\psi(\rho_1) + (1-s)\psi(\rho_2))$ if and only if ψ has one of the forms: (1) $\rho \mapsto \sigma_0$ for any $\rho \in \mathcal{S}(H)$; (2) $\psi(\mathcal{P}ur(H)) = \{Q_1, Q_2\}$; (3) $\rho \mapsto \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$, where σ_0 is a pure state on K and $M : H \to K$ is an injective linear or conjugate linear operator. For multipartite systems, we also give a structure theorem for maps that preserve separable pure states and strict convex combinations. These results allow us to characterize injective (local) quantum measurements and answer some conjectures proposed in [8].

1. Introduction

In the theory of quantum information, a state is a positive operator of trace 1 acting on a complex Hilbert space H. Denote by $\mathcal{S}(H)$ and $\mathcal{P}ur(H)$ respectively the set of all states and the set of all pure states ($i.e.$ rank-1 projections) on H. In quantum information theory we deal, in general, with multipartite systems. The underlying space H of a multipartite composite quantum system is a tensor product of underlying spaces H_i of its subsystems, that is $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. If $n = 2$, the system is called a bipartite system. The definition of multipartite separability was introduced in [13] as a natural extension of the notion of separability in bipartite case [14]. Let us denote the set of all states in an n-partite system by $\mathcal{S}(H_1 \otimes \cdots \otimes H_n)$. In the case $\dim H < \infty$, a state $\rho \in \mathcal{S}(H_1 \otimes \cdots \otimes H_n)$ is said to be (fully) separable if it admits a representation of the form

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \cdots \otimes \rho_i^{(n)},$$

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where $p_i > 0$ with $\sum_i p_i = 1$ and $\rho_i^{(k)} \in \mathcal{S}(H_k)$. Otherwise, ρ is said to be entangled. Denote by respectively $\mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$ and $\mathcal{P}ur_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$ the set of all separable states and the set of all separable pure states on $H_1 \otimes H_2 \otimes \cdots \otimes H_n$. It is obvious that

$$\begin{aligned} & \mathcal{P}ur_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) = \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2) \otimes \cdots \otimes \mathcal{P}ur(H_n) \\ = & \ \{P_1 \otimes P_2 \otimes \cdots \otimes P_n : P_i \in \mathcal{P}ur(H_i), i = 1, 2, \ldots, n\}. \end{aligned}$$

The theory of maps on the set of states plays an important role in quantum computation and quantum information science. It is important to understand, characterize, and construct different classes of maps on states. For instance, all quantum channels and quantum operations are completely positive linear maps; in quantum error correction, one has to construct the recovery map for a given channel; to study the entanglement of states, one constructs NCP (non completely positive) positive maps and entanglement witnesses. Many researchers pay their attention to the problem of characterizing the maps on the states; ref. [1, 2, 3, 6].

Recently, authors also pay their attention to characterizing convex combination preserving maps. Recall that a map ϕ between convex sets is said to be (strict) convex combination preserving if, for any $\rho, \sigma \in \mathcal{S}(H)$ and $t \in [0, 1]$ ($t \in (0, 1)$), there is some s with $0 \leq s \leq 1$ ($0 < s < 1$) such that $\phi(t\rho + (1-t)\sigma) = s\phi(\rho) + (1-s)\phi(\sigma)$. It is obvious that ϕ preserves (strict) convex combination if and only if $\phi([\rho, \sigma]) \subseteq [\phi(\rho), \phi(\sigma)]$ ($\phi((\rho, \sigma)) \subseteq (\phi(\rho), \phi(\sigma))$), where $[A, B]$ stands for the closed (open) line segment joint A and B, that is, $[A, B] = \{tA + (1-t)B : 0 \leq t \leq 1\}$ ($(A, B) = [A, B] \setminus \{A, B\}$, here we define $(\rho, \rho) = \{\rho\}$).

We remark that the maps preserve strict convex combinations are closely related to quantum measurements. In quantum mechanics a fine-grained quantum measurement is described by a collection $\{M_m\}$ of measurement operators acting on the Hilbert space H corresponding to the system satisfying $\sum_m M_m^* M_m = I$; ref. for example, [4]. Let M_j be a measurement operator. If the state of the quantum system is $\rho \in \mathcal{S}(H)$ before the measurement, then the state after the measurement is $\frac{M_j \rho M_j^*}{\mathrm{Tr}(M_j \rho M_j^*)}$ whenever $M_j \rho M_j^* \neq 0$. If M_j is fixed, we get a measurement map ϕ_j defined by $\phi_j(\rho) = \frac{M_j \rho M_j^*}{\mathrm{Tr}(M_j \rho M_j^*)}$ from the convex subset $\mathcal{S}_{M_j}(H) = \{\rho : M_j \rho M_j^* \neq 0\}$ of the (convex) set $\mathcal{S}(H)$ into $\mathcal{S}(H)$. If M_j is invertible (injective), then $\phi_j : \mathcal{S}(H) \to \mathcal{S}(H)$ is bijective (injective) and will be called an invertible (injective) measurement map. Observe that a measurement map is strict convex combination preserving. Another feature of the quantum measurement maps is that they send pure states to pure states. Thus, it is a natural and basic task to study the following

Problem (1°) *How to characterize the maps on convex subsets of states that preserve pure states and strict convex combinations, and reveal the relation of such maps to the quantum measurements.*

This topic was firstly attacked by [5]. It is shown that a bijective map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$, $\dim H \geq 2$, is (strict) convex combination preserving if and only if ϕ is an invertible quantum measurement map or the composition of transpose and an invertible quantum measurement map, that is, ϕ has one of the following forms $\rho \mapsto \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$ or $\rho \mapsto \frac{M\rho^t M^*}{\mathrm{Tr}(M\rho^t M^*)}$, where M is an invertible operator and ρ^t is the transpose of ρ with respect to an arbitrarily fixed orthonormal basis. Note that ϕ is a bijective (strict) convex combination preserver implies that $\phi(\mathcal{P}ur(H)) = \mathcal{P}ur(H)$.

Next let us consider the multipartite systems. Every local quantum measurement map ϕ of the form $\phi(\rho) = \frac{(S\otimes T)\rho(S\otimes T)^*}{\mathrm{Tr}((S\otimes T)\rho(S\otimes T)^*)}$ is a strict convex combination preserving map from $\{\rho \in \mathcal{S}(H_1 \otimes H_2) : (S \otimes T)\rho(S \otimes T)^* \neq 0\}$ into $\mathcal{S}(H_1 \otimes H_2)$, which sends separable pure states into separable pure states as well as sends separable states in the convex subset $\{\rho \in \mathcal{S}_{\rm sep}(H_1 \otimes H_2) : (S\otimes T)\rho(S\otimes T)^* \neq 0\}$ into separable states, where $S \in \mathcal{B}(H_1)$ and $T \in \mathcal{B}(H_2)$. This fact makes it interesting to study the inverse problem, that is,

Problem (2°) *How to characterize (strict) convex combination preserving maps between the convex subsets of $\mathcal{S}(H_1 \otimes H_2)$ which send separable pure states to separable pure states and reveal the relation of such maps to the local quantum measurements.*

For the case $\dim H_1 \otimes H_2 < \infty$, if a map preserves separable pure states, then it preserves separable states. So, without loss of generality, we may $\mathcal{S}(H_1 \otimes H_2)$ by $\mathcal{S}_{\rm sep}(H_1 \otimes H_2)$ in Problem (2°) if we consider only the finite dimensional systems.

Recall that a state $\rho \in \mathcal{S}(H_1 \otimes H_2)$ is called a product state if $\rho = \rho_1 \otimes \rho_2$ for some $\rho_i \in \mathcal{S}(H_i)$, $i = 1, 2$. Let $\phi : \mathcal{S}_{\rm sep}(H_1 \otimes H_2) \to \mathcal{S}_{\rm sep}(H_1 \otimes H_2)$ be a bijective map. In [8], based on the work of [5], it is shown that, if ϕ is (strict) convex combination preserving and if $\phi(P_1 \otimes P_2)$ is a product state for any $P_i \in \mathcal{S}(H_i)$ with $\mathrm{rank}P_i = 1$ and $\mathrm{rank}P_j = 2$ $(1 \leq i \neq j \leq 2)$, then ϕ is a composition of an invertible local quantum measurement (i.e., the map of the form $\rho \mapsto \frac{(S\otimes T)\rho(S\otimes T)^*}{\mathrm{Tr}((S\otimes T)\rho(S\otimes T)^*)}$ with S, T invertible) and some of the following maps: the transpose, the partial transpose and the swap. It is conjectured in [8] that

Conjecture (3°) *The additional assumption*

$$\begin{aligned} &\phi(P_1 \otimes P_2) \quad \textit{is a product state for any } P_i \in \mathcal{S}(H_i) \text{ with} \\ &\mathrm{rank}P_i = 1 \quad \textit{and rank} P_j = 2, \quad (1 \leq i \neq j \leq 2) \end{aligned} \tag{1.1}$$

is superfluous.

Note that the local quantum measurement maps, the transpose with respect to a product basis, the partial transposes and the swap are all maps preserve separable pure states and strict convex combinations. Moreover, they also preserve product states, that is, they send

the state of the form $\rho \otimes \sigma$ into the state of the form $\rho' \otimes \sigma'$. It was also conjectured in [8, Conjecture 5.2] that

Conjecture (4°) *Let $\phi : \mathcal{S}_{\rm sep}(H_1 \otimes H_2) \to \mathcal{S}_{\rm sep}(H_1 \otimes H_2)$ be an injective map. If ϕ is convex combination preserving and separable pure state preserving, then ϕ preserves product states.*

The present paper is to continue the study of above mentioned questions for finite dimensional systems. In this paper, a map is said to be essentially a (local) quantum measurement if it is a (local) quantum measurement or a composition of a (local) quantum measurement with any one of the following maps: the transpose, the partial transpose, and the swap.

In Section 2 we discuss the problem (1°) and characterize successfully the maps ψ from $\mathcal{S}(H)$ into $\mathcal{S}(K)$ with $2 \leq \dim H < \infty$ that preserve strict convex combinations and pure states, which generalizes the main result in [5]. We show that such maps ψ have one of the following three forms: (1) ψ is contractive to a pure state, i.e, there exists a pure state $Q \in \mathcal{P}ur(K)$ such that $\psi(\rho) = Q$ for all $\rho \in \mathcal{S}(H)$; (2) there exist distinct pure states $Q_i \in \mathcal{P}ur(K)$, $i = 1, 2$ such that $\psi(\mathcal{P}ur(H)) = \{Q_1, Q_2\}$, and $\psi(\mathcal{S}(H)) \subseteq [Q_1, Q_2]$; (3) $\dim H \leq \dim K$ and there exists an injective operator $M \in \mathcal{B}(H, K)$ such that $\psi(\rho) = \frac{M\rho M^*}{{\rm Tr}(M\rho M^*)}$ for all $\rho \in \mathcal{S}(H)$, or $\psi(\rho) = \frac{M\rho^t M^*}{{\rm Tr}(M\rho^t M^*)}$ for all $\rho \in \mathcal{S}(H)$, where A^t is the transpose of A with respect to an arbitrarily fixed orthonormal basis of H, that is, ψ is essentially an injective quantum measurement (see Theorem 2.5). Note that, by our result, if ψ is strict convex combination preserving and pure state preserving and if ψ is not continuous, then ψ must have the form (2), and an example of such map is given (Remark 2.7).

In Section 3 we study the problem (2°) for bipartite systems. Based on the results obtained in Section 2, we are able to give a structure theorem of maps $\psi : \mathcal{S}_{\rm sep}(H_1 \otimes H_2) \to \mathcal{S}_{\rm sep}(K_1 \otimes K_2)$ that preserve strict convex combinations and separable pure states. We show that such maps can have ten possible forms (Theorem 3.2). Consequently, if the range of ψ is non-collinear or a singleton, then ψ sends product states to product states (Corollary 3.3), which answers the conjecture (4°) raised in [8] affirmatively; and moreover, if the range of ψ also contains a state σ so that its reductions ${\rm Tr}_1(\sigma)$ and ${\rm Tr}_2(\sigma)$ both have rank ≥ 2, then ψ is essentially an injective local quantum measurement (Corollary 3.4). Particularly, we show that the additional condition Eq.(1.1) is not necessary for the main result of [8], this answers another conjecture in [8], that is, (3°) mentioned above.

Section 4 is a brief discussion of the same topic for multipartite systems. The similar structure theorem is valid for maps $\psi : \mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{S}_{\rm sep}(K_1 \otimes K_2 \otimes \cdots \otimes K_n)$ that preserve strict convex combinations and separable pure states, but with more complicated

expressions. Particularly, if the range of ψ is non-collinear or a singleton, then ψ sends product states to product states; and moreover, if the range of ψ contains a state σ so that each reduction $\mathrm{Tr}^i(\sigma)$ has rank ≥ 2, then ψ is essentially a local injective quantum measurement (Theorem 4.1, Corollary 4.2).

Section 5 is a short conclusion.

2. Maps preserving pure states and strict convex combinations

In this section we discuss the problem (1°) and characterize the maps between the convex sets of quantum states that send pure states to pure states and preserve the strict convex combinations.

We start by giving a simple lemma.

Lemma 2.1. *Let $\{Q_1, Q_2, \cdots, Q_r\}$ be a linearly independent set of rank one projections acting on a Hilbert space H. If $\sum_{i=1}^r t_i Q_i = P_0$ is a projection for some $t_i > 0$,$i = 1, 2, \ldots, r$, then $\{Q_i\}_{i=1}^r$ are orthogonal and $t_i = 1$ for all i.*

Proof. There are unit vectors $x_i \in H$ such that $Q_i = x_i \otimes x_i$. If $\sum_{i=1}^r t_i Q_i = P_0$ is a projection for some $t_i > 0$, then $P_0^2 = P_0 \geq 0$, and thus $(\sum_{i=1}^r t_i Q_i)^2 = \sum_{i=1}^r {t_i}^2 x_i \otimes x_i + \sum_{i\neq j} t_i t_j \langle x_j, x_i\rangle x_i \otimes x_j = \sum_{i=1}^r t_i x_i \otimes x_i$. This implies that

$$\sum_{i\neq j} t_i t_j \langle x_j, x_i\rangle x_i \otimes x_j = \sum_{i=1}^r \left(t_i - t_i^2\right) x_i \otimes x_i \tag{2.1}$$

Since $\{x_i\}_{i=1}^r$ is a linearly independent set, there exist $\{y_j\}_{j=1}^r \subset H$, such that $\langle y_j, x_i\rangle = \delta_{ij}$. Letting the two sides of Eq.(2.1) act at y_j $(1 \leq j \leq r)$ respectively, we get $\sum_{i\neq j} t_i t_j \langle x_j, x_i\rangle x_i = (t_j - t_j^2)x_j$. It follows that $t_j - t_j^2 = 0$ and $t_i t_j \langle x_j, x_i\rangle = 0$. Thus $t_j = 1$ for all j and $\langle x_j, x_i\rangle = 0$ for any $i \neq j$, that is, $\{Q_i\}_{i=1}^r$ is an orthogonal set of rank one projections. □

Let $\mathbf{H}_m$ be the real linear space of all $m \times m$ Hermitian matrices and let $\mathcal{P}_m$ be the set of all rank-1 $m \times m$ projection matrices. The next lemma comes from [3] which can be viewed as a characterization of linear preservers of pure states. Also, ref. [9] for infinite dimensional case.

Lemma 2.2. Suppose $\phi : \mathbf{H}_m \to \mathbf{H}_n$ is a linear map satisfying $\phi(\mathcal{P}_m) \subseteq \mathcal{P}_n$. Then one of the following holds:

(i) *There is $Q \in \mathcal{P}_n$ such that $\phi(A) = \mathrm{Tr}(A)Q$ for all $A \in \mathbf{H}_m$.*

(ii) *$m \leq n$ and there is a $U \in M_{n\times m}$ with $U^*U = I_m$ such that $\phi(A) = UAU^*$ for all $A \in \mathbf{H}_m$, or $\phi(A) = UA^tU^*$ for all $A \in \mathbf{H}_m$.*

The following lemma is the main result in [12], which gives a characterization of strict convex combination preserving maps in terms of linear ones.

Lemma 2.3. *Let X and Y be real linear spaces and $D \subseteq X$ a nonempty convex subset. Assume that $\phi : D \to Y$ is a strict convex combination preserving map such that $\phi(D)$ is non-collinear (i.e., $\phi(D)$ contains a nondegenerate triangle). Then, there exist a linear transformation $A : X \to Y$, a linear functional $f : X \to \mathbb{R}$, a vector $y_0 \in Y$, and a scalar $b \in \mathbb{R}$ such that*

$$f(x) + b > 0 \quad \textit{for all} \quad x \in D$$

and

$$\phi(x) = \frac{Ax + y_0}{f(x) + b} \quad \textit{for all} \quad x \in D.$$

By using of Lemma 2.1 and Lemma 2.3 we can prove the following lemma, which is also crucial for proving our main result.

Lemma 2.4. *Let H be a complex Hilbert space with $2 \leq \dim H = r < \infty$ and $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ be a pure state and strict convex combination preserving map. If $\phi(\frac{1}{r}I) = \frac{1}{r}I$ and $\mathrm{ran}\phi$ is non-collinear, then ϕ is affine.*

Proof. As ϕ preserves pure states and strict convex combinations, by Lemma 2.1, $\phi(\frac{1}{r}I) = \frac{1}{r}I$ implies that ϕ maps orthogonal pure states to orthogonal pure states. Also, by Lemma 2.3, ϕ is strict convex combination preserving and $\mathrm{ran}\phi$ is non-collinear together imply that ϕ has the form $\phi(\rho) = \frac{\Gamma(\rho)+D}{f(\rho)+d}$ for any $\rho \in \mathcal{S}(H)$, where $\Gamma : \mathcal{B}_{sa}(H) \to \mathcal{B}_{sa}(H)$ is a linear transformation, $f : \mathcal{B}_{\rm sa}(H) \to \mathbb{R}$ is a linear functional, $D \in \mathcal{B}_{\rm sa}(H)$ and $d \in \mathbb{R}$ with $f(\rho)+d > 0$ for all $\rho \in \mathcal{S}(H)$, $\mathcal{B}_{\rm sa}(H)$ is the real linear space of all self-adjoint operators in $\mathcal{B}(H)$. Since $\dim H < \infty$, Γ and f are continuous. It follows that ϕ is continuous. To prove the lemma, we consider two cases of $\dim H > 2$ and $\dim H = 2$ respectively.

Case 1. $\dim H > 2$.

We will show that f is a constant on $\mathcal{S}(H)$, that is, there is a real number a such that $f(\rho) = a$ for all $\rho \in \mathcal{S}(H)$.

For any normalized orthogonal basis $\{e_i\}_{i=1}^r$ of H, let $P_i = e_i \otimes e_i$. We first claim that $f(e_i \otimes e_i) = f(e_j \otimes e_j)$ for any i and j. Since ϕ preserves pure states, there is a pure state $Q_i = x_i \otimes x_i$ such that

$$x_i \otimes x_i = Q_i = \phi(P_i) = \frac{\Gamma(e_i \otimes e_i) + D}{f(e_i \otimes e_i) + d}.$$

So

$$\Gamma(e_i \otimes e_i) + D = (f(e_i \otimes e_i) + d)(x_i \otimes x_i).$$

As $\phi(\frac{I}{r}) = \frac{I}{r}$ and $\frac{I}{r} = \frac{1}{r}\sum_{i=1}^{r} e_i \otimes e_i$, we have

$$\frac{I}{r} = \phi(\frac{1}{r}\sum_{i=1}^{r} e_i \otimes e_i) = \frac{\Gamma(\sum_{i=1}^{r}\frac{1}{r}e_i \otimes e_i) + D}{f(\sum_{i=1}^{r}\frac{1}{r}e_i \otimes e_i) + d} = \frac{\sum_{i=1}^{r}\frac{1}{r}\Gamma(e_i \otimes e_i) + r\frac{1}{r}D}{\sum_{i=1}^{r}\frac{1}{r}f(e_i \otimes e_i) + r\frac{1}{r}d},$$

that is,

$$\frac{I}{r} = \frac{\frac{1}{r}(\sum_{i=1}^{r}(\Gamma(e_i \otimes e_i) + D))}{\frac{1}{r}(\sum_{i=1}^{r}(f(e_i \otimes e_i) + d))} = \frac{\sum_{i=1}^{r}(\Gamma(e_i \otimes e_i) + D)}{\sum_{i=1}^{r}(f(e_i \otimes e_i) + d)}.$$

Let $A_i = \Gamma(e_i \otimes e_i) + D$ and $a_i = f(e_i \otimes e_i) + d$. Then $A_i = a_i Q_i$ and the above equation becomes to

$$I = r(\frac{A_1 + A_2 + \cdots + A_r}{a_1 + a_2 + \cdots + a_r}) = r(\frac{a_1Q_1 + a_2Q_2 + \cdots + a_rQ_r}{a_1 + a_2 + \cdots + a_r}).$$

Applying Lemma 2.1, we see that $\frac{ra_i}{a_1+a_2+\cdots+a_r} = 1$ for each $i = 1, 2, \ldots, r$ and hence

$$a_1 = a_2 = \cdots = a_r = \frac{a_1 + a_2 + \cdots + a_r}{r}.$$

This implies that there is some scalar a such that $f(e_i \otimes e_i) = a$ holds for all i. Now for arbitrary unit vectors x, $y \in H$, as $\dim H > 2$, there is a unit vector $z \in H$ such that $z \in [x, y]^\perp$. It follows from what proved above that $f(x \otimes x) = f(z \otimes z) = f(y \otimes y)$. So $f(x \otimes x) = a$ for all unit vectors $x \in H$. Since each state is a convex combination of pure states, by the linearity of f, we get that $f(\rho) = a$ holds for every state ρ. Therefore, we have

$$\phi(\rho) = \frac{\Gamma(\rho) + D}{a + d}$$

holds for all ρ. Then by the linearity of Γ, it is clear that ϕ is affine, i.e., for any states ρ, σ and scalar λ with $0 \leq \lambda \leq 1$, $\phi(\lambda\rho + (1-\lambda)\sigma) = \lambda\phi(\rho) + (1-\lambda)\phi(\sigma)$.

Case 2. $\dim H = 2$.

By fixing an orthonormal basis of H we may identify $\mathcal{S}(H)$ with $\mathcal{S}_2$, the convex set of 2×2 positive matrices with the trace 1. Then $\phi : \mathcal{S}_2 \to \mathcal{S}_2$ is a map preserving pure states and strict convex combinations satisfying $\phi(\frac{1}{2}I_2) = \frac{1}{2}I_2$. Let us identify $\mathcal{S}_2$ with the Bloch ball representation $(\mathbb{R}^3)_1 = \{(x, y, z)^t \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ by the following way. Let $\pi : (\mathbb{R}^3)_1 \to \mathcal{S}_2$ be the map defined by

$$(x, y, z)^t \longmapsto \frac{1}{2}I_2 + \frac{1}{2}\begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix}.$$

π is a bijective affine isomorphism. Note that $v = (x, y, z)^t$ satisfies $x^2 + y^2 + z^2 = 1$ if and only if the corresponding matrix $\pi(v)$ is a pure state, and $0 = (0, 0, 0)^t$ if and only if the corresponding matrix is $\pi(0) = \frac{1}{2}I$. The map $\phi : \mathcal{S}_2 \to \mathcal{S}_2$ induces a map $\hat{\phi} : (\mathbb{R}^3)_1 \to (\mathbb{R}^3)_1$ by the following equation

$$\phi(\rho) = \pi(\hat{\phi}(\pi^{-1}(\rho))).$$

Since ϕ is pure state and strict convex combination preserving and continuous, and π is an affine isomorphism, it is easily checked that the map $\hat{\phi}$ is strict convex combination preserving and maps the surface of $(\mathbb{R}^3)_1$ into the surface of $(\mathbb{R}^3)_1$. Since $\phi(\frac{1}{2}I) = \frac{1}{2}I$, we have that $\hat{\phi}((0,0,0)^t) = (0,0,0)^t$. It is also clear that the range of $\hat{\phi}$ is non-collinear.

Now applying Lemma 2.3 to $\hat{\phi}$, there exists a linear transformation $L : \mathbb{R}^3 \to \mathbb{R}^3$, a linear functional $f : \mathbb{R}^3 \to \mathbb{R}$, a vector $u_0 \in \mathbb{R}^3$ and a scalar $r \in \mathbb{R}$ such that $f((x,y,z)^t) + r > 0$ and

$$\hat{\phi}((x,y,z)^t) = \frac{L((x,y,z)^t) + u_0}{f((x,y,z)^T) + r}$$

for each $(x,y,z)^T \in (\mathbb{R}^3)_1$. Since $\hat{\phi}((0,0,0)^T) = (0,0,0)^t$, we have $u_0 = 0$ and $r > 0$. Furthermore, the linearity of f implies that there are real scalars r_1, r_2, r_3 such that $f((x,y,z)^t) = r_1x + r_2y + r_3z$. We claim that $r_1 = r_2 = r_3 = 0$ and hence $f = 0$. If not, then there is a vector $(x_0,y_0,z_0)^T$ satisfying $x_0^2 + y_0^2 + z_0^2 = 1$ such that $f((x_0,y_0,z_0)^T) = r_1x_0 + r_2y_0 + r_3z_0 \neq 0$. It follows that

$$1 = \|\hat{\phi}((x_0,y_0,z_0)^t)\| = \|\frac{L((x_0,y_0,z_0)^t)}{r_1x_0 + r_2y_0 + r_3z_0 + r}\|,$$

and thus

$$\|L((x_0,y_0,z_0)^t)\| = r_1x_0 + r_2y_0 + r_3z_0 + r.$$

Similarly

$$\|L((-x_0,-y_0,-z_0)^t)\| = -r_1x_0 - r_2y_0 - r_3z_0 + r.$$

By the linearity of L we have $r_1x_0 + r_2y_0 + r_3z_0 + r = -r_1x_0 - r_2y_0 - r_3z_0 + r$. Hence $r_1x_0 + r_2y_0 + r_3z_0 = 0$, a contradiction. So, we have $f = 0$, and thus $\hat{\phi} = \frac{L}{r}$ is linear. Now it is clear that ϕ is affine as π is an affine isomorphism. $\square$

The following is the main result of this section which gives a characterization of maps preserving pure states and strict convex combinations.

Theorem 2.5. *Let H, K be complex Hilbert spaces with $2 \leq \dim H < \infty$ and $\mathcal{S}(H), \mathcal{S}(K)$ the convex sets of all states on H, K, respectively. Let $\psi : \mathcal{S}(H) \to \mathcal{S}(K)$ be a map. Then ψ preserves pure states and strict convex combination (that is, $\psi(\mathcal{P}ur(H)) \subseteq \mathcal{P}ur(K)$ and $\psi((\rho,\sigma)) \subseteq (\psi(\rho),\psi(\sigma))$ for any $\rho,\sigma \in \mathcal{S}(H)$) if and only if one of the following holds:*

(1) *There exists $\sigma_0 \in \mathcal{P}ur(K)$ such that $\psi(\rho) = \sigma_0$ for all $\rho \in \mathcal{S}(H)$.*

(2) *There exist distinct pure states $Q_i \in \mathcal{P}ur(K)$, $i = 1,2$ such that $\psi(\mathcal{P}ur(H)) = \{Q_1, Q_2\}$, and a map $h : \mathcal{S}(H) \to [0,1]$ such that, for any $\rho_1, \rho_2 \in \mathcal{S}(H)$ and any $t \in (0,1)$, $h(t\rho_1 + (1-t)\rho_2) = sh(\rho_1) + (1-s)h(\rho_2)$ for some $s \in (0,1)$, and $\psi(\rho) = h(\rho)Q_1 + (1-h(\rho))Q_2$ for all $\rho \in \mathcal{S}(H)$.*

(3) $\dim H \leq \dim K$ *and there exists an injective operator* $M \in \mathcal{B}(H,K)$ *such that* $\psi(\rho) = \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$ *for all* $\rho \in \mathcal{S}(H)$*, or* $\psi(\rho) = \frac{M\rho^t M^*}{\mathrm{Tr}(M\rho^t M^*)}$ *for all* $\rho \in \mathcal{S}(H)$*, where* A^t *is the transpose of* A *with respect to an arbitrarily fixed orthonormal basis of* H*.*

We remark here that the form (3) can be restated as:

($3'$) $\dim H \leq \dim K$ *and there exists an injective linear or conjugate linear operator* $M : H \to K$ *such that* $\psi(\rho) = \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$ *for all* $\rho \in \mathcal{S}(H)$*.*

This statement is more convenient some times.

The following corollary is immediate, which essentially gives a characterization of injective quantum measurement maps or the transpose of an injective quantum measurement. We say that a map ψ is open line segment preserving if $\psi((\rho,\sigma)) = (\psi(\rho),\psi(\sigma))$ for any ρ,σ.

Corollary 2.6. *Let* H,K *be complex Hilbert spaces with* $2 \leq \dim H < \infty$ *and* $\psi : \mathcal{S}(H) \to \mathcal{S}(K)$ *be a map. Then the following statements are equivalent.*

(1) ψ *is strict convex combination preserving with* $\psi(\mathcal{P}ur(H)) \subseteq \mathcal{P}ur(K)$ *and non-collinear range.*

(2) ψ *is open line segment preserving with* $\psi(\mathcal{P}ur(H)) \subseteq \mathcal{P}ur(K)$ *and non-collinear range.*

(3) $\dim H \leq \dim K$ *and there exists an injective operator* $M \in \mathcal{B}(H,K)$ *such that* $\psi(\rho) = \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$ *for all* $\rho \in \mathcal{S}(H)$*, or* $\psi(\rho) = \frac{M\rho^t M^*}{\mathrm{Tr}(M\rho^t M^*)}$ *for all* $\rho \in \mathcal{S}(H)$*, where* A^t *is the transpose of* A *with respect to an arbitrarily fixed orthonormal basis of* H*.*

Particularly, if ψ is bijective, then, by [5, Lemma 2.1], we have $\psi(\mathcal{P}ur(H)) = \mathcal{P}ur(K)$. Also the surjectivity of ψ implies the surjectivity of M. Thus the above corollary is a generalization of the main result in [5] for finite dimensional case.

Remark 2.7. In the cases (1) and (3) of Theorem 2.5, the map ψ is continuous. However, in the case (2), ψ is not continuous and may have erratic form. For example, Assume H is of dimension 2. Let Q_1, Q_2 be two distinct pure states on K. Divide $\mathcal{P}ur(H)$ into two disjoint parts $\mathcal{P}ur(H) = \mathcal{P}_1 \cup \mathcal{P}_2$ with the property $P \in \mathcal{P}_1 \Leftrightarrow P^\perp \in \mathcal{P}_2$ and define $\psi(P) = Q_1$ if $P \in \mathcal{P}_1$; $\psi(P) = Q_2$ if $P \in \mathcal{P}_2$; $\psi(tP_1 + (1-t)P_2) = \frac{1}{2}(Q_1 + Q_2)$ if $P_i \in \mathcal{P}_i$, $i = 1,2$, and $t \in (0,1)$, where Q_1, Q_2 are any distinct pure states on K. Then, $\psi : \mathcal{S}(H) \to \mathcal{S}(K)$ is strict convex combination preserving and $\psi(\mathcal{P}ur(H)) \subset \mathcal{P}ur(K)$. ψ has the form (2) in Theorem 2.5.

Now let us start to prove the main result of this section.

Proof of Theorem 2.5.

If ψ has the form (1), (2) or (3), it is clear that ψ is pure states and strict convex combination preserving. Conversely, assume that ψ is pure states and strict convex combination preserving. We will show that ψ has one of the forms stated in (1), (2) and (3).

Assume $\dim H = m < \infty$.

As $\frac{1}{m}I \in \mathcal{S}(H)$, where I is the identity on H, $\psi(\frac{1}{m}I)$ is positive with trace 1. So $\psi(\frac{1}{m}I) = \frac{RR^*}{\mathrm{Tr}(RR^*)}$ for some bounded linear operator R from H into K.

Claim 1. $\dim \mathrm{ran}R \leq m$ and $\mathrm{ran}\psi(\rho) \subseteq \mathrm{ran}R$ holds for all $\rho \in \mathcal{S}(H)$.

Firstly we will show that $\mathrm{ran}\psi(P) \subseteq \mathrm{ran}R$ holds for any pure state $P \in \mathcal{S}(H)$. Let $P = P_1$. There exist pure states $\{P_2, \cdots, P_m\}$ such that $\{P_1, P_2, \cdots, P_m\}$ is an orthogonal set satisfying $P_1 + P_2 + \cdots + P_m = I$. Since ψ is strict convex combination preserving, there are $p_i \in (0,1)$ $(i = 1, 2, \cdots, m)$ with $\sum_{i=1}^m p_i = 1$ such that

$$\frac{RR^*}{\mathrm{Tr}(RR^*)} = \psi(\frac{1}{m}I) = \psi(\frac{1}{m}\sum_{i=1}^m P_i) = \sum_{i=1}^m p_i \psi(P_i).$$

Note that $\psi(P_i)$s are rank-1 projections by the assumption. It follows that $\dim \mathrm{ran}R \leq \sum_{i=1}^m \dim \mathrm{ran}\psi(P_i) \leq m$ and, for any i, we have $0 \leq \psi(P_i) \leq p_i^{-1}\frac{RR^*}{\mathrm{Tr}(RR^*)}$. Hence $\mathrm{ran}\psi(P_i) \subseteq \mathrm{ran}R$ for all i. Particularly, $\mathrm{ran}\psi(P) = \mathrm{ran}\psi(P_1) \subseteq \mathrm{ran}R$.

For any $\rho \in \mathcal{S}(H)$, let $\rho = \sum_{i=1}^m t_i P_i$ be its spectral resolution. As $t_i \geq 0$, $\sum_i t_i = 1$ and ψ is strict convex combination preserving, there are $s_i \in [0,1]$ with $\sum_i s_i = 1$ and $s_i = 0$ if $t_i = 0$ such that $\psi(\rho) = \sum_{i=1}^m s_i \psi(P_i)$. Now it is clear that $\mathrm{ran}\psi(\rho) \subseteq \mathrm{ran}R$ because $\mathrm{ran}\psi(P_i) \subseteq \mathrm{ran}R$ for all $i = 1, 2, \ldots, m$.

As a result, if R is of rank-1, then it is clear that ψ has the form (1) in Theorem 2.5.

So, in the sequel, we assume that $\mathrm{rank}(R) = r \geq 2$. Thus we can define the map $\phi : \mathcal{S}(H) \to \mathcal{S}(H)$ by

$$\phi(\rho) = \frac{R^{[-1]}\psi(\rho)R^{[-1]*}}{\mathrm{Tr}(R^{[-1]}\psi(\rho)R^{[-1]*})}, \tag{2.2}$$

where $R^{[-1]}$ is the Moore-Penrose generalized inverse of R. It is clear that $\phi(\mathcal{P}ur(H)) \subseteq \mathcal{P}ur(H)$ and ϕ preserves strict convex combination. Write $\phi(\frac{1}{m}I_m) = \frac{Q_0}{\mathrm{Tr}(Q_0)}$, where $Q_0 = R^{[-1]}R$. This implies that Q_0 is a projection with $\mathrm{rank}Q_0 = r \leq m$. It follows that, there exists an orthonormal set $\{e_1, \ldots, e_r\} \subset H$ such that $\sum_{i=1}^r \phi(P_i) = Q_0$, where $P_i = e_i \otimes e_i$. Let $H_1 = \mathrm{span}\{e_1, \ldots, e_r\}$ and $K_1 = Q_0(H)$. Then $\dim H_1 = \dim K_1 = r$ and $\widetilde{\phi} = \phi|_{\mathcal{S}(H_1)} : \mathcal{S}(H_1) \to \mathcal{S}(K_1)$ is a strict convex combination preserver sending pure states to pure states. As $\widetilde{\phi}(\frac{1}{r}I_{H_1}) = \sum_{i=1}^r q_i\phi(P_i) = SS^*$ is an invertible state on K_1, it induces a strict convex combination preserver $\widehat{\phi} : \mathcal{S}(H_1) \to \mathcal{S}(H_1)$ satisfying $\widehat{\phi}(\mathcal{P}ur(H_1)) \subseteq \mathcal{P}ur(H_1)$ and $\widehat{\phi}(\frac{1}{r}I_{H_1}) = \frac{1}{r}I_{H_1}$, where $\widehat{\phi}$ is defined by

$$\widehat{\phi}(\rho) = \frac{S^{-1}\widetilde{\phi}(\rho)S^{-1*}}{\mathrm{Tr}(S^{-1}\widetilde{\phi}(\rho)S^{-1*})}. \tag{2.3}$$

It follows that $\widehat{\phi}$ maps orthogonal pure states to orthogonal ones.

Claim 2. If $r \geq 3$, then there is a unitary operator $U : H_1 \to H_1$ such that either $\widehat{\phi}(\rho) = U\rho U^*$ for every $\rho \in \mathcal{S}(H_1)$ or $\widehat{\phi}(\rho) = U\rho^t U^*$ for every $\rho \in \mathcal{S}(H_1)$.

If $r \geq 3$, then $\widehat{\phi}(\mathcal{S}(H_1))$ contains at least 3 rank one projections which are orthogonal to each other and hence non-collinear. So, by Lemma 2.4, $\widehat{\phi}$ is affine, that is $\widehat{\phi}(t\rho + (1-t)\sigma) = t\widehat{\phi}(\rho) + (1-t)\widehat{\phi}(\sigma)$ holds for any $\rho, \sigma \in \mathcal{S}(H_1)$ and $t \in [0,1]$. Thus $\widehat{\phi}$ is injective and can be extended to an injective linear map from $\mathcal{B}_{\rm sa} \cong \mathbf{H}_r$ into $\mathcal{B}_{\rm sa} \cong \mathbf{H}_r$.

Now by Lemma 2.2, there is a unitary operator $U : H_1 \to H_1$ such that either $\widehat{\phi}(\rho) = U\rho U^*$ for every $\rho \in \mathcal{S}(H_1)$ or $\widehat{\phi}(\rho) = U\rho^t U^*$ for every $\rho \in \mathcal{S}(H_1)$.

If P and Q are projections and $PQ = 0$, we say that P and Q are orthogonal, denoted by $P \perp Q$. $P^\perp$ stands for $I - P$.

Claim 3. If $r = 2$, then either

(i) there exist $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathcal{P}ur(H_1)$ satisfying $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}ur(H_1)$ and $P, I_2 - P$ can not be in the same $\mathcal{P}_i$; and there exist $Q_1, Q_2 \in \mathcal{P}ur(H_1)$ with $Q_1 \perp Q_2$ such that $\widehat{\phi}(\mathcal{P}_i) = \{Q_i\}$, $i = 1, 2$, and $\widehat{\phi}(\mathcal{S}(H_1) \subseteq [Q_1, Q_2]$.

(ii) there exists a unitary operator $U : H_1 \to H_1$ such that either $\widehat{\phi}(\rho) = U\rho U^*$ for every $\rho \in \mathcal{S}(H_1)$ or $\widehat{\phi}(\rho) = U\rho^t U^*$ for every $\rho \in \mathcal{S}(H_1)$.

If there exist $Q_1, Q_2, Q_3 \in \widehat{\phi}(\mathcal{S}(H_1))$ such that they are non-collinear, then Lemma 2.4 is applicable. It follows that $\widehat{\phi}$ is affine and can be extended to a linear or conjugate linear map, still denoted by $\widehat{\phi}$ from $\mathbf{H}_2$ into $\mathbf{H}_2$, which is unital and rank-1 projection preserving. Therefore, use Lemma 2.2 we see that (ii) is true.

Assume that $\widehat{\phi}(\mathcal{S}(H_1)$ is collinear; then there exist $P_1, P_2 \in \mathcal{S}(H_1)$ such that $\widehat{\phi}(P_i) = Q_i$, $Q_1 + Q_2 = I_2$ and $\widehat{\phi}(\mathcal{S}(H_1) \subseteq [Q_1, Q_2]$. It entails that $\widehat{\phi}(\mathcal{P}ur(H_1)) = \{Q_1, Q_2\}$. Let $\mathcal{P}_i = \widehat{\phi}^{-1}(\{Q_i\})$, $i = 1, 2$. Then, $\mathcal{P}_1 \cap \mathcal{P}_2 = \emptyset$ and $\mathcal{P}_1 \cup \mathcal{P}_2 = \mathcal{P}ur(H_1)$. If $P \in \mathcal{P}_1$, then

$$\frac{1}{2}\widehat{\phi}(P) + \frac{1}{2}\widehat{\phi}(P^\perp) = \widehat{\phi}(\frac{1}{2}(P + P^\perp)) = \frac{1}{2}I_2.$$

So $\widehat{\phi}(P^\perp) = Q_2$, and $P^\perp \in \mathcal{P}_2$. Now, it is clear that (i) holds.

Claim 4. If $r = m \geq 3$, then $\psi(\rho) = \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$ for all $\rho \in \mathcal{S}(H)$; or $\psi(\rho) = \frac{M\rho^t M^*}{\mathrm{Tr}(M\rho^t M^*)}$ for all $\rho \in \mathcal{S}(H)$, where $M : H \to K$ is an injective linear operator. So, ψ has the form (3) of Theorem 2.5.

In fact, in this case we have $\widehat{\phi} = \widetilde{\phi} = \phi$. By Claim 2, there is a unitary $U : H \to H$ such that $\phi(\rho) = U\rho U^*$ for all ρ or $\phi(\rho) = U\rho^t U^*$ for all ρ. Let $M = RU$. Then $M : H \to K$ is injective and the claim holds.

By use of Claim 4, the following claim is obvious.

Claim 5. If $r = m = 2$, then either ψ has the form (3) or has the form (2) of Theorem 2.5.

Claim 6. If $2 = r < m$, then ψ has the form (2).

As $r=2$, $\widehat{\phi}$ has two possible forms (i) and (ii) stated in Claim 3.

If $\widehat{\phi}$ has the form (i), then there exist distinct pure states Q_1, Q_2 on K such that $\psi(\mathcal{P}ur(H)) = \{Q_1, Q_2\}$. It is clear that, in this case, we have $\psi(\rho) \in [Q_1, Q_2]$ for every $\rho \in \mathcal{S}(H)$, that is, ψ is of the form (2) stated in Theorem 2.5.

We assert that the case (ii) does not occur. If $\widehat{\phi}$ takes form (ii), then there exists an orthogonal set of pure states $\{P_1 = e_1 \otimes e_1, \ldots, P_m = e_m \otimes e_m\}$ such that $\operatorname{ran}(\psi) = \mathcal{S}(K_1)$, where $K_1 = \operatorname{span}\{u_1, u_2\}$ with $u_i \otimes u_i = \psi(e_i \otimes e_i)$, $i = 1, 2$. Thus ψ is continuous when restricted on $\mathcal{S}(H_1)$ with $H_1 = \operatorname{span}\{e_1, e_2\}$. Note that $\psi(P_3) = \psi(P_1)$ or $\psi(P_2)$, say $\psi(P_3) = \psi(P_1) = Q_1$. Let $P(\alpha, \beta) = (\alpha e_1 + \beta e_2) \otimes (\alpha e_1 + \beta e_2) \in \mathcal{P}ur(H_1)$, where $\alpha, \beta \in \mathbb{C}$ satisfying $|\alpha|^2 + |\beta|^2 = 1$. Then, $P(\alpha, \beta)$ is continuous and hence $\psi(P(\alpha, \beta))$ is continuous in α, β. Since $\{P(\alpha, \beta), P(\bar{\beta}, -\bar{\alpha}), P_3, \ldots, P_m\}$ is still a complete orthogonal set of rank-1 projections, one of them must be $\psi(P_3) = Q_1$ and another be Q_2. It follows that the range of $\psi(P(\alpha, \beta))$ can take at most two distinct value and thus must be a constant function. However, $\psi(P(1, 0)) = \psi(P_1) = Q_1$ and $\psi(P(0, 1)) = \psi(P_2) = Q_2$, a contradiction. So, this case can not occur, finishing the proof of Claim 6.

Claim 7. The case $3 \leq r < m$ can not occur.

On the contrary, suppose $m > r \geq 3$. Then by Lemma 2.3, ϕ is continuous. Choose any orthogonal set of rank one projections $\{P_i = x_i \otimes x_i\}_{i=1}^m$ satisfying $\sum_{i=1}^m P_i = I$. Then there exists $p_i > 0$ with $\sum_{i=1}^m p_i = 1$ such that $\phi(\frac{1}{m} I) = \phi(\frac{1}{m} \sum_{i=1}^m P_i) = \sum_{i=1}^m p_i \phi(P_i) = \frac{1}{r} Q_0$. It follows that m rank one projections $\phi(P_1), \cdots, \phi(P_m)$ are linearly dependent. Without loss of generality, assume $\{\phi(P_1), \cdots, \phi(P_r)\}$ is linearly independent. Then for any $j > r$, there exists $i_j \leq r$ such that $\phi(P_j) = \phi(P_{i_j})$. So there exist $q_i > 0$, $(1 \leq i \leq r)$, such that $\phi(\frac{1}{m} I) = \sum_{i=1}^r q_i \phi(P_i)$. By Lemma 2.1, we obtain that $q_i = \frac{1}{r}$ and $\{\phi(P_i)\}_{i=1}^r$ is an orthogonal set of rank-1 projections. Consequently we obtain that, for any two orthogonal rank one projections $Q_1 = y_1 \otimes y_1$, $Q_2 = y_2 \otimes y_2$ on H, either $\phi(Q_1) = \phi(Q_2)$ or $\phi(Q_1) \perp \phi(Q_2)$. Let $Q_1' = (\alpha y_1 + \beta y_2) \otimes (\alpha y_1 + \beta y_2)$, $Q_2' = (\bar{\beta} y_1 - \bar{\alpha} y_2) \otimes (\bar{\beta} y_1 - \bar{\alpha} y_2)$, where $|\alpha|^2 + |\beta|^2 = 1$. Clearly $Q_1' \perp Q_2'$. We assert that:

$$\phi(Q_1) = \phi(Q_2) \Rightarrow \phi(Q_1') = \phi(Q_2') \quad \text{and} \quad \phi(Q_1) \perp \phi(Q_2) \Rightarrow \phi(Q_1') \perp \phi(Q_2'). \tag{2.4}$$

To see this, let $f(\alpha, \beta) = \phi(Q_1')$, $g(\alpha, \beta) = \phi(Q_2')$ and let $h(\alpha, \beta) = \|f(\alpha, \beta) - g(\alpha, \beta)\|$. As ϕ is continuous, we see that f, g and h are continuous in α, β. Also note that $h(\alpha, \beta) \in \{0, 1\}$ for any (α, β). Hence, if $\phi(Q_1) = \phi(Q_2)$, then $h(1, 0) = 0$, which forces $h(\alpha, \beta) \equiv 0$ and consequently, $f(\alpha, \beta) = g(\alpha, \beta)$ for all α, β; if $\phi(Q_1) \perp \phi(Q_2)$, then $h(\alpha, \beta) \equiv h(0, 1) = 1$, which implies that $f(\alpha, \beta) \perp g(\alpha, \beta)$ for all α, β. So the assertion (2.4) is true.

Now for the chosen orthogonal set $\{P_i\}_{i=1}^m$, as $3 \le r < m$, by what proved above, we can rearrange the order of $\{\phi(P_i)\}_{i=1}^m$ so that $\{\phi(P_1), \ldots, \phi(P_r)\}$ is an orthogonal set. Then $\phi(P_1) + \cdots + \phi(P_r) = Q_0$, $\phi(P_{r+j})$ equals to $\phi(P_i)$ for some i with $1 \le i \le r$.

Assume that there exist two distinguished projections in $\{\phi(P_{r+1}), \ldots, \phi(P_m)\}$, say $\phi(P_{r+1}) \neq \phi(P_{r+2})$. Let $P'_{r+1} = (\alpha x_{r+1} + \beta x_{r+2}) \otimes (\alpha x_{r+1} + \beta x_{r+2})$ and $P'_{r+2} = (\bar{\beta} x_{r+1} - \bar{\alpha} x_{r+2}) \otimes (\bar{\beta} x_{r+1} - \bar{\alpha} x_{r+2})$ with $|\alpha|^2 + |\beta|^2 = 1$. By Eq.(2.4), $f(\alpha, \beta) \perp g(\alpha, \beta)$, where $f(\alpha, \beta) = \phi(P'_{r+1})$ and $g(\alpha, \beta) = \phi(P'_{r+2})$. Since $\{P_1, \ldots, P_r, P'_{r+1}, P'_{r+2}, \ldots, P_m\}$ is still orthogonal, we see that $f(\alpha, \beta) \in \{\phi(P_1), \ldots, \phi(P_r)\}$. The continuity of f then implies that $f(\alpha, \beta) \equiv \phi(P_{i_0})$ for some $1 \le i_0 \le r$. Similarly, $g(\alpha, \beta) \equiv \phi(P_{i_1})$ for some $1 \le i_1 \le r$. Since f and g has the same range, we must have $i_0 = i_1$, but this contradicts to $f(1, 0) \perp g(1, 0)$.

Therefore, we may assume that $\phi(P_{r+1}) = \cdots = \phi(P_m) = \phi(P_1)$. Now Let $P'_1 = (\alpha x_1 + \beta x_2) \otimes (\alpha x_1 + \beta x_2)$ and $P'_2 = (\bar{\beta} x_1 - \bar{\alpha} x_2) \otimes (\bar{\beta} x_1 - \bar{\alpha} x_2)$ with $|\alpha|^2 + |\beta|^2 = 1$. Denote $f(\alpha, \beta) = \phi(P'_1)$ and $g(\alpha, \beta) = \phi(P'_2)$. Then by Eq.(2.4) again we have $f(\alpha, \beta) \perp g(\alpha, \beta)$. Note that $f(1, 0) = \phi(P_1) = \phi(P_{r+1}) \in \{f(\alpha, \beta), g(\alpha, \beta)\}$ for any (α, β). This entails $\{f(\alpha, \beta), g(\alpha, \beta)\} = \{\phi(P_1), \phi(P_2)\}$ for any (α, β). So it follows from the continuity of f and g that $f(\alpha, \beta) \equiv \phi(P_1)$ and $g(\alpha, \beta) \equiv \phi(P_2)$, contradicting to the fact that f and g has the same range. So the claim is true.

Combining Claims 1-7, we see that ψ preserves pure state and strict convex combination will imply that ψ takes one of the form (1), (2) and (3), completing the proof of Theorem 2.5. $\square$

3. Maps preserving separable pure states and strict convex combinations: bipartite systems

This section is devoted to investigating the problem (2°) and giving a structure theorem of maps preserve separable pure states and strict convex combinations for bipartite systems. Using this structure theorem we are able to answer the conjectures (3°) and (4°) proposed in [8] for finite dimensional case.

We start by a simple lemma which may be found in [10].

Lemma 3.1 *Let H be a complex Hilbert space of any dimension and $T \in \mathcal{B}(H)$ a finite rank operator. Then $\frac{1}{\mathrm{rank} T}\|T\|_{\mathrm{Tr}}^2 \le \|T\|_2^2 \le \|T\|_{\mathrm{Tr}}^2$, where $\|T\|_{\mathrm{Tr}}$ and $\|T\|_2$ are respectively the trace-norm and the Hilbert-Schmidt norm of T.*

Recall that $\mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2)$ and $\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$ stand respectively for the convex set of all separable states and the set of all separable pure states in bipartite system $H_1 \otimes H_2$. The following is our main result in this section.

Theorem 3.2 *Let H_1, H_2, K_1, K_2 be complex Hilbert spaces with $2 \leq \dim H_i < \infty$, $i = 1, 2$. Let $\psi : \mathcal{S}_{\rm sep}(H_1 \otimes H_2) \to \mathcal{S}_{\rm sep}(K_1 \otimes K_2)$ be a map. If ψ preserves separable pure states and strict convex combinations, then one of the following statements holds.*

(1) *There exists $R_1 \otimes R_2 \in \mathcal{P}ur(K_1) \otimes \mathcal{P}ur(K_2)$ such that*

$$\psi(A \otimes B) = R_1 \otimes R_2$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$.

(2) $\dim H_1 \leq \dim K_1$, *there exist $R_2 \in \mathcal{P}ur(K_2)$ and injective $M_1 \in \mathcal{B}(H_1, K_1)$ such that*

$$\psi(A \otimes B) = \frac{M_1 A M_1^*}{\mathrm{Tr}(M_1 A M_1^*)} \otimes R_2 \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2)$$

or

$$\psi(A \otimes B) = \frac{M_1 A^t M_1^*}{\mathrm{Tr}(M_1 A^t M_1^*)} \otimes R_2 \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2).$$

(3) $\dim H_2 \leq \dim K_2$, *there exist $R_1 \in \mathcal{P}ur(K_1)$ and injective $M_2 \in \mathcal{B}(H_2, K_2)$ such that*

$$\psi(A \otimes B) = R_1 \otimes \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)} \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2)$$

or

$$\psi(A \otimes B) = R_1 \otimes \frac{M_2 B^t M_2^*}{\mathrm{Tr}(M_2 B^t M_2^*)} \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2).$$

(4) $\dim K_1 \geq \dim H_2$, *there exist $R_2 \in \mathcal{P}ur(K_2)$ and injective $M_1 \in \mathcal{B}(H_2, K_1)$ such that*

$$\psi(A \otimes B) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes R_2 \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2)$$

or

$$\psi(A \otimes B) = \frac{M_1 B^t M_1^*}{\mathrm{Tr}(M_1 B^t M_1^*)} \otimes R_2 \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2).$$

(5) $\dim H_1 \leq \dim K_2$, *there exist $R_1 \in \mathcal{P}ur(K_1)$ and injective $M_2 \in \mathcal{B}(H_1, K_2)$ such that*

$$\psi(A \otimes B) = R_1 \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)} \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2)$$

or

$$\psi(A \otimes B) = R_1 \otimes \frac{M_2 A^t M_2^*}{\mathrm{Tr}(M_2 A^t M_2^*)} \text{ for all } A \in \mathcal{S}(H_1) \text{ and } B \in \mathcal{S}(H_2).$$

(6) $\dim H_i \leq \dim K_i$, $i = 1, 2$, *there exist injective $M_1 \in \mathcal{B}(H_1, K_1)$ and $M_2 \in \mathcal{B}(H_2, K_2)$ such that*

$$\psi(A \otimes B) = \frac{M_1 \Psi_1(A) M_1^*}{\mathrm{Tr}(M_1 \Psi_1(A) M_1^*)} \otimes \frac{M_2 \Psi_2(B) M_2^*}{\mathrm{Tr}(M_2 \Psi_2(B) M_2^*)}$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$*, where* $\Psi_i : \mathcal{B}(H_i) \to \mathcal{B}(H_i)$*,* $i = 1, 2$*, is the identity, or the transpose with respect to an arbitrarily fixed orthonormal basis.*

(7) $\dim H_1 \leq \dim K_2$ *and* $\dim H_2 \leq \dim K_1$*, there exist injective* $M_1 \in \mathcal{B}(H_2, K_1)$ *and* $M_2 \in \mathcal{B}(H_1, K_2)$ *such that*

$$\psi(A \otimes B) = \frac{M_1 \Psi_2(B) M_1^*}{\mathrm{Tr}(M_1 \Psi_2(B) M_1^*)} \otimes \frac{M_2 \Psi_1(A) M_2^*}{\mathrm{Tr}(M_2 \Psi_1(A) M_2^*)}$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$*, where* $\Psi_i : \mathcal{B}(H_i) \to \mathcal{B}(H_i)$*,* $i = 1, 2$*, is the identity, or the transpose with respect to an arbitrarily fixed orthonormal basis.*

(8) $\min\{\dim H_1, \dim H_2\} \leq \dim K_1$*, there exist* $R_2 \in \mathcal{P}ur(K_2)$ *and a strict convex combination preserving map* $\varphi_1 : \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2) \to \mathcal{S}(K_1)$ *such that*

$$\psi(A \otimes B) = \varphi_1(A \otimes B) \otimes R_2$$

for all $A \in \mathcal{S}(H_1)$ *and* $B \in \mathcal{S}(H_2)$*. Moreover,* φ_1 *satisfies that for each* $P \otimes Q \in \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$*,* $\varphi_1(P \otimes Q) = \frac{M_P Q M_P^*}{\mathrm{Tr}(M_P Q M_P^*)} = \frac{N_Q P N_Q^*}{\mathrm{Tr}(N_Q P N_Q^*)}$ *for some injective linear or conjugate linear (may not synchronously) operators* $M_P : H_2 \to K_1$ *and* $N_Q : H_1 \to K_1$*.*

(9) $\min\{\dim H_1, \dim H_2\} \leq \dim K_2$*, there exist* $R_1 \in \mathcal{P}ur(K_1)$ *and a strict convex combination preserving map* $\varphi_2 : \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2) \to \mathcal{S}(K_2)$ *such that*

$$\psi(A \otimes B) = R_1 \otimes \varphi_2(A \otimes B)$$

for all $A \in \mathcal{S}(H_1)$ *and* $B \in \mathcal{S}(H_2)$*. Moreover,* φ_2 *satisfies that, for each* $P \otimes Q \in \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$*,* $\varphi_2(P \otimes Q) = \frac{M_P Q M_P^*}{\mathrm{Tr}(M_P Q M_P^*)} = \frac{N_Q P N_Q^*}{\mathrm{Tr}(N_Q P N_Q^*)}$*, for some injective linear or conjugate linear (may not synchronously) operators* $M_P : H_2 \to K_2$ *and* $N_Q : H_1 \to K_2$*.*

(10) *There exist* $P_i' \in \mathcal{P}ur(K_1)$ *and* $Q_i' \in \mathcal{P}ur(K_2)$*,* $i = 1, 2$ *such that* $\psi(\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)) = \{P_1' \otimes Q_1', P_2' \otimes Q_2'\}$*, and* $\mathrm{ran}(\psi) \subseteq [P_1' \otimes Q_1', P_2' \otimes Q_2']$*.*

Proof. Suppose $\dim H_1 = m$, $\dim H_2 = n$. If the range of ψ is collinear, then it is clear that either (1) holds or (10) holds. So, in the sequel, we assume that the range of ψ is non-collinear. Denote by $\mathcal{B}_{\mathrm{sa}}(H)$ the real linear space of all self-adjoint operators on the Hilbert space H. Consider the partial traces $\mathrm{Tr}_1 : \mathcal{B}_{\mathrm{sa}}(K_1 \otimes K_2) \to \mathcal{B}_{\mathrm{sa}}(K_2)$ and $\mathrm{Tr}_2 : \mathcal{B}_{\mathrm{sa}}(K_1 \otimes K_2) \to \mathcal{B}_{\mathrm{sa}}(K_1)$ on $\mathcal{B}_{\mathrm{sa}}(K_1 \otimes K_2) \equiv \mathcal{B}_{\mathrm{sa}}(K_1) \otimes \mathcal{B}_{\mathrm{sa}}(K_2)$ defined by $\mathrm{Tr}_1(A \otimes B) = (\mathrm{Tr} A)B$ and $\mathrm{Tr}_2(A \otimes B) = (\mathrm{Tr} B)A$. Clearly, Tr_1 and Tr_2 are linear maps. Define two maps $\phi_1 : (\mathcal{S}(H_1), \mathcal{S}(H_2)) \to \mathcal{S}(K_1)$ and $\phi_2 : (\mathcal{S}(H_1), \mathcal{S}(H_2)) \to \mathcal{S}(K_2)$ by

$$\phi_1(A, B) = \mathrm{Tr}_2(\psi(A \otimes B)) \text{ and } \phi_2(A, B) = \mathrm{Tr}_1(\psi(A \otimes B)).$$

Notice that

$$\psi(P \otimes Q) = \phi_1(P, Q) \otimes \phi_2(P, Q) \text{ for all } P \in \mathcal{P}ur(H_1) \text{ and } Q \in \mathcal{P}ur(H_2).$$

Fix a $Q \in \mathcal{P}ur(H_2)$; then the maps $\phi_1(\cdot, Q) : \mathcal{S}(H_1) \to \mathcal{S}(K_1)$ and $\phi_2(\cdot, Q) : \mathcal{S}(H_1) \to \mathcal{S}(K_2)$ are both strict convex combination preserving and $\phi_1(\mathcal{P}ur(H_1), Q) \subseteq \mathcal{P}ur(K_1)$ while $\phi_2(\mathcal{P}ur(H_1), Q) \subseteq \mathcal{P}ur(K_2)$. Therefore, applying Theorem 2.5 to $\phi_1(\cdot, Q)$ and $\phi_2(\cdot, Q)$, respectively, we get that, for $i = 1, 2$, either

(i) there exists pure state $R_{iQ} \in \mathcal{P}ur(K_i)$ such that $\phi_i(A, Q) = R_{iQ}$ for all $A \in \mathcal{S}(H_1)$;

or

(ii) there are pure states $R^{(i)}_{1Q}, R^{(i)}_{2Q} \in \mathcal{P}ur(K_i)$ and a strict convex combination preserving map $h_{iQ} : \mathcal{S}(H_1) \to [0, 1]$ such that $\phi_i(\mathcal{P}ur(H_1), Q) = \{R^{(i)}_{1Q}, R^{(i)}_{2Q}\}$ and $\phi_i(A, Q) = h_{iQ}(A)R^{(i)}_{1Q} + (1 - h_{iQ}(A))R^{(i)}_{2Q}$ for all $A \in \mathcal{S}(H_1)$;

or

(iii) there exists an injective linear or conjugate linear operator $M_{iQ} : H_1 \to K_i$ such that $\phi_i(A, Q) = \frac{M_{iQ}AM^*_{iQ}}{\mathrm{Tr}(M_{iQ}AM^*_{iQ})}$ for all $A \in \mathcal{S}(H_1)$.

As ψ is strict convex combination preserving and ranψ is non-collinear, Lemma 2.3 is applicable. Thus we have $\psi(\rho) = \frac{\Gamma(\rho)+D}{f(\rho)+d}$, where $\Gamma : \mathcal{B}_{\mathrm{sa}}(H_1 \otimes H_2) \to \mathcal{B}_{\mathrm{sa}}(K_1 \otimes K_2)$ is a linear map, $D \in \mathcal{B}_{\mathrm{sa}}(K_1 \otimes K_2)$ is an operator, $f : \mathcal{B}_{\mathrm{sa}}(H_1 \otimes H_2) \to \mathbb{R}$ is a linear functional and $d \in \mathbb{R}$ with $f(\rho) + d > 0$ for all $\rho \in \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2)$. As both H_1 and H_2 are finite dimensional, Γ, f are continuous. It follows that ψ is continuous. So both ϕ_1 and ϕ_2 are continuous on $\mathcal{S}(H_1)$. These facts will be used frequently.

Let us first consider the map ϕ_1.

Claim 1. Either $\phi_1(\cdot, Q)$ has the form (i) for all $Q \in \mathcal{P}ur(H_2)$ or $\phi_1(\cdot, Q)$ has the form (iii) for all $Q \in \mathcal{P}ur(H_2)$.

As mentioned above, for any fixed pure state $Q \in \mathcal{P}ur(H_2)$, $\phi_1(\cdot, Q)$ takes one of the forms (i)-(iii). As ϕ_1 is continuous, by Remark 2.7 we see that $\phi_1(\cdot, Q)$ can not have the form (ii) for any $Q \in \mathcal{P}ur(H_2)$. Thus, for any Q, $\phi_1(\cdot, Q)$ takes the form (i) or the form (iii).

Furthermore, we will show that either $\phi_1(\cdot, Q)$ has the form (i) for all $Q \in \mathcal{P}ur(H_2)$ or $\phi_1(\cdot, Q)$ has the form (iii) for all $Q \in \mathcal{P}ur(H_2)$.

To do this, for any $A \in \mathcal{S}(H_1)$ so that $\mathrm{rank}(A) \geq 2$, define $F_A : \mathcal{P}ur(H_2) \to \mathbb{R}$ by $F_A(Q) = \|\phi_1(A, Q)\|_2$, where $\|\cdot\|_2$ is the Hilbert-Schmidt norm. Notice that when $\phi_1(\cdot, Q)$ takes the form (i), then $F_A(Q) = \|\phi_1(A, Q)\|_2 = 1$; when $\phi_1(\cdot, Q)$ takes the form (iii), then $\phi_1(A, Q) = \frac{M_{1Q}AM^*_{1Q}}{\mathrm{Tr}(M_{1Q}AM^*_{1Q})}$ and $F_A(Q) = \|\phi_1(A, Q)\|_2 = \frac{\|M_{1Q}AM^*_{1Q}\|_2}{\|M_{1Q}A^{\frac{1}{2}}\|_2^2} < 1$ as rank$A \geq 2$ and M_{1Q} is injective.

If there exist two distinct $Q_1, Q_2 \in \mathcal{P}ur(H_2)$, such that $\phi_1(\cdot, Q_1)$ has the form (i) while $\phi_1(\cdot, Q_2)$ has the form (iii), that is, $\phi_1(\cdot, Q_1) = R_{1Q_1}$, $\phi_1(\cdot, Q_2) = \frac{M_{1Q_2}(\cdot)M^*_{1Q_2}}{\mathrm{Tr}(M_{1Q_2}(\cdot)M^*_{1Q_2})}$. Let $Q_1 = x \otimes x$ and $Q_2 = y \otimes y$ with unit vectors $x, y \in H_2 \cong \mathbb{C}^n$. Note that x and y are linearly

independent. For any $t \in [0,1]$, define

$$Q(t) = \frac{1}{\|x+t(y-x)\|^2}(x+t(y-x)) \otimes (x+t(y-x)) \in \mathcal{P}ur(H_2).$$

Then, $Q(t)$ is continuous in t, $Q(0) = Q_1$, $Q(1) = Q_2$, and $\phi_1(\cdot, Q(t))$ has the form (i) or (iii) for any t. Fix an $A \in \mathcal{S}(H_1)$ with $\mathrm{rank}A \geq 2$ and let $t_0 = \max\{t \in [0,1] : F_A(Q(t)) = 1\}$. Then $F_A(Q(t_0)) = 1$ and so $\phi_1(\cdot, Q(t_0))$ has the form (i). Thus $\|\phi_1(A, Q(t_0))\|_2 = 1$ for any $A \in \mathcal{S}(H_1)$. For any $1 \geq t > t_0$, $\phi_1(\cdot, Q(t))$ has the form (iii). Thus there exist $\{t_n\}$, $t_n > t_0$, such that $\phi_1(\cdot, Q(t_n))$ has the form (iii) and $t_n \to t_0$. Then by Lemma 3.1, for any given sufficient small $\varepsilon > 0$, there exist $\{A_{t_n}\} \subseteq \mathcal{S}(H_1)$ with $\mathrm{rank}A_{t_n} = 2$, such that $\frac{1}{2} \leq \|\phi_1(A_{t_n}, Q(t_n))\|_2^2 \leq \frac{1}{2} + \varepsilon < 1$. The reason of the existence of $\{A_{t_n}\}$ for each n is that, we can find rank-2 operator $B_{t_n} \in \mathrm{ran}\phi_1(\cdot, Q_{t_n})$ such that $\frac{1}{2} \leq \|B_{t_n}\|_2^2 \leq \frac{1}{2} + \varepsilon < 1$, thus there exists $A_{t_n} \in \mathcal{S}(H_1)$ such that $B_{t_n} = \frac{M_{t_n} A_{t_n} M_{t_n}^*}{\mathrm{Tr}(M_{t_n} A_{t_n} M_{t_n}^*)}$. As M_{t_n} is injective, we see that $\mathrm{rank}A_{t_n} = 2$. Now, since $\mathcal{S}(H_1)$ is a compact set, $\{A_{t_n}\} \subseteq \mathcal{S}(H_1)$ has a convergent subsequence $\{A_{t_{n_i}}\} \subseteq \{A_{t_n}\}$, say $A_{t_{n_i}} \to A_0$ as $t_{n_i} \to t_0$. Then the continuity of ψ entails that $\|\phi_1(A_{t_{n_i}}, Q(t_{n_i}))\|_2^2 \to \|\phi_1(A_0, Q(t_0))\|_2^2$. But this is a contradiction because $\|\phi_1(A_0, Q(t_0))\|_2^2 = 1$ and $\|\phi_1(A_{t_{n_i}}, Q(t_{n_i}))\|_2^2 \leq \frac{1}{2} + \varepsilon < 1$. Thus either $\phi_1(\cdot, Q)$ has the form (i) for all $Q \in \mathcal{P}ur(H_2)$ or $\phi_1(\cdot, Q)$ has the form (iii) for all $Q \in \mathcal{P}ur(H_2)$.

Similarly, we have

Claim 1′. Either $\phi_2(\cdot, Q)$ has the form (i) for all $Q \in \mathcal{P}ur(H_2)$ or $\phi_2(\cdot, Q)$ has the form (iii) for all $Q \in \mathcal{P}ur(H_2)$.

Claim 2. One of the following holds:

(a) For all $Q \in \mathcal{P}ur(H_2)$, both $\phi_1(\cdot, Q)$ and $\phi_2(\cdot, Q)$ have the form (i).

(b) For all $Q \in \mathcal{P}ur(H_2)$, $\phi_1(\cdot, Q)$ has the form (i) and $\phi_2(\cdot, Q)$ has the form (iii).

(c) For all $Q \in \mathcal{P}ur(H_2)$, $\phi_1(\cdot, Q)$ has the form (iii) and $\phi_2(\cdot, Q)$ has the form (i).

We need only to check that, for all $Q \in \mathcal{P}ur(H_2)$, $\phi_1(\cdot, Q)$ and $\phi_2(\cdot, Q)$ can not have the form (iii) simultaneously. Suppose, on the contrary, there exists $Q_0 \in \mathcal{P}ur(H_2)$ such that both $\phi_1(\cdot, Q_0)$ and $\phi_2(\cdot, Q_0)$ are of the form (iii). Then there exist injective linear or conjugate linear operators $M_{1Q_0} : H_1 \to K_1$ and $M_{2Q_0} : H_1 \to K_2$ such that $\phi_i(A, Q_0) = \frac{M_{iQ_0} A M_{iQ_0}^*}{\mathrm{Tr}(M_{iQ_0} A M_{iQ_0}^*)}$ for all $A \in \mathcal{S}(H_1)$, $i = 1, 2$. Thus, we must have $\dim H_1 \leq \min\{\dim K_1, \dim K_2\}$ and

$$\begin{aligned} \psi(P \otimes Q_0) = & \quad \phi_1(P, Q_0) \otimes \phi_2(P, Q_0) = \left(\frac{M_{1Q_0} P M_{1Q_0}^*}{\mathrm{Tr}(M_{1Q_0} P M_{1Q_0}^*)}\right) \otimes \left(\frac{M_{2Q_0} P M_{2Q_0}^*}{\mathrm{Tr}(M_{2Q_0} P M_{2Q_0}^*)}\right) \\ = & \quad \frac{(M_{1Q_0} \otimes M_{2Q_0})(P \otimes P)(M_{1Q_0} \otimes M_{2Q_0})^*}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P \otimes P)(M_{1Q_0} \otimes M_{2Q_0})^*)} \end{aligned}$$

for all $P \in \mathcal{P}ur(H_1)$. Particularly, take $P_1 = e_1 \otimes e_1$, $P_2 = e_2 \otimes e_2$, $P_3 = \frac{1}{2}(e_1 \otimes e_1 + e_1 \otimes e_2 + e_2 \otimes e_1 + e_2 \otimes e_2)$ and $P_4 = \frac{1}{2}(e_1 \otimes e_1 - e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_2)$. Then $P_1 + P_2 = P_3 + P_4$ and so $P_1 \otimes Q_0 + P_2 \otimes Q_0 = P_3 \otimes Q_0 + P_4 \otimes Q_0$. As ψ is strict convex combination preserving,

there exist $s_1 \in (0,1)$ and $s_2 \in (0,1)$, such that

$$\begin{aligned} & \psi(\tfrac{1}{2}P_1 \otimes Q_0 + \tfrac{1}{2}P_2 \otimes Q_0) = s_1\psi(P_1 \otimes Q_0) + (1-s_1)\psi(P_2 \otimes Q_0) \\ = \quad & s_1 \tfrac{(M_{1Q_0} \otimes M_{2Q_0})(P_1 \otimes P_1)(M_{1Q_0} \otimes M_{2Q_0})^*}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_1 \otimes P_1)(M_{1Q_0} \otimes M_{2Q_0})^*)} + (1-s_1) \tfrac{(M_{1Q_0} \otimes M_{2Q_0})(P_2 \otimes P_2)(M_{1Q_0} \otimes M_{2Q_0})^*}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_2 \otimes P_2)(M_{1Q_0} \otimes M_{2Q_0})^*)} \end{aligned}$$

and

$$\begin{aligned} & \psi(\tfrac{1}{2}P_3 \otimes Q_0 + \tfrac{1}{2}P_4 \otimes Q_0) = s_2\psi(P_3 \otimes Q_0) + (1-s_2)\psi(P_4 \otimes Q_0) \\ = \quad & s_2 \tfrac{(M_{1Q_0} \otimes M_{2Q_0})(P_3 \otimes P_3)(M_{1Q_0} \otimes M_{2Q_0})^*}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_3 \otimes P_3)(M_{1Q_0} \otimes M_{2Q_0})^*)} + (1-s_2) \tfrac{(M_{1Q_0} \otimes M_{2Q_0})(P_4 \otimes P_4)(M_{1Q_0} \otimes M_{2Q_0})^*}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_4 \otimes P_4)(M_{1Q_0} \otimes M_{2Q_0})^*)}. \end{aligned}$$

It follows that there exist $a_i \neq 0$, $i = 1, 2, 3, 4$, such that

$$a_1 P_1 \otimes P_1 + a_2 P_2 \otimes P_2 = a_3 P_3 \otimes P_3 + a_4 P_4 \otimes P_4.$$

In fact, $a_1 = \frac{s_1}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_1 \otimes P_1)(M_{1Q_0} \otimes M_{2Q_0})^*)}$, $a_2 = \frac{1-s_1}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_2 \otimes P_2)(M_{1Q_0} \otimes M_{2Q_0})^*)}$, $a_3 = \frac{s_2}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_3 \otimes P_3)(M_{1Q_0} \otimes M_{2Q_0})^*)}$, $a_4 = \frac{1-s_2}{\mathrm{Tr}((M_{1Q_0} \otimes M_{2Q_0})(P_4 \otimes P_4)(M_{1Q_0} \otimes M_{2Q_0})^*)}$. As $P_4 = P_1 + P_2 - P_3$, then $0 = a_1 P_1 \otimes P_1 + a_2 P_2 \otimes P_2 - a_3 P_3 \otimes P_3 - a_4 P_4 \otimes P_4 = P_1 \otimes a_1 P_1 + P_2 \otimes a_2 P_2 - P_3 \otimes a_3 P_3 - P_1 \otimes a_4 P_4 - P_2 \otimes a_4 P_4 + P_3 \otimes a_4 P_4 = P_1 \otimes (a_1 P_1 - a_4 P_4) + P_2 \otimes (a_2 P_2 - a_4 P_4) - P_3 \otimes (a_3 P_3 - a_4 P_4)$. As P_1, P_2 and P_3 are linearly independent, by [7], we have $a_1 P_1 - a_4 P_4 = 0$, $a_2 P_2 - a_4 P_4 = 0$ and $a_3 P_3 - a_4 P_4 = 0$, which is a contradiction. So the claim is true.

Similarly, one can check that

Claim 3. One of the following holds:

(a$'$) For all $P \in \mathcal{P}ur(H_1)$, both $\phi_1(P,\cdot)$ and $\phi_2(P,\cdot)$ have the form (i).

(b$'$) For all $P \in \mathcal{P}ur(H_1)$, $\phi_1(P,\cdot)$ has the form (i) and $\phi_2(P,\cdot)$ has the form (iii).

(c$'$) For all $P \in \mathcal{P}ur(H_1)$, $\phi_1(P,\cdot)$ has the form (iii) and $\phi_2(P,\cdot)$ has the form (i).

Claim 4. (a) and (a$'$) can not hold simultaneously.

In fact, if (a) and (a$'$) hold, that is, for all $Q \in \mathcal{P}ur(H_2)$, we have $\phi_i(A,Q) = R_{iQ}$, and, for all $P \in \mathcal{P}ur(H_1)$, we have $\phi_i(P,B) = R_{iP}$. Fix $P_0 \in \mathcal{P}ur(H_1)$ and $Q_0 \in \mathcal{P}ur(H_2)$. Then we get

$$\phi_i(P,Q) = \phi_i(P,Q_0) = \phi_i(P_0,Q_0) = R_i.$$

Therefore, $\psi(P \otimes Q) = \phi_1(P,Q) \otimes \phi_2(P,Q) = R_1 \otimes R_2$ for all $P \otimes Q \in \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$. We know that for any $A \otimes B \in \mathcal{S}(H_1) \otimes \mathcal{S}(H_2)$, there exist $\{A_i\}_{i=1}^m \subseteq \mathcal{P}ur(H_1)$, $\{B_j\}_{j=1}^n \subseteq \mathcal{P}ur(H_2)$, $\{a_i\}_{i=1}^m \subseteq [0,1]$, $\{b_j\}_{j=1}^n \subseteq [0,1]$ satisfying $\sum_{i=1}^m a_i = 1$, $\sum_{j=1}^n b_j = 1$, such that $A = \sum_{i=1}^m a_i A_i$, $B = \sum_{j=1}^n b_j B_j$. Then, there exist $c_{ij} \geq 0$ with $\sum_{i,j} c_{ij} = 1$ such that $\psi(A \otimes B) = \sum_{i,j} c_{ij} \psi(A_i \otimes B_j) = R_1 \otimes R_2$, this contradicts to the assumption that the range of ψ is non-collinear. So the Claim 4 is true.

Claim 5. If (a) and (b$'$) hold, then ψ has the form (3), that is, there exist $R_1 \in \mathcal{P}ur(K_1)$ and injective linear or conjugate linear operator $M_2 : H_2 \to K_2$ such that

$$\psi(A \otimes B) = R_1 \otimes \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. It is clear that $\dim H_2 \leq \dim K_2$.

In this case, for any $P \otimes Q \in \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$ we have

$$\psi(P \otimes Q) = \phi_1(P, Q) \otimes \phi_2(P, Q) = R_{1Q} \otimes R_{2Q} = R_{1P} \otimes \frac{M_{2P} Q M_{2P}^*}{\mathrm{Tr}(M_{2P} Q M_{2P}^*)},$$

which implies that $R_{1Q} = R_{1P}$ is independent of P, Q, and $\frac{M_{2P} Q M_{2P}^*}{\mathrm{Tr}(M_{2P} Q M_{2P}^*)} = R_{2Q}$. Thus for any fixed $Q \in \mathcal{P}ur(H_2)$, $\frac{M_{2P_1} Q M_{2P_1}^*}{\mathrm{Tr}(M_{2P_1} Q M_{2P_1}^*)} = \frac{M_{2P_2} Q M_{2P_2}^*}{\mathrm{Tr}(M_{2P_2} Q M_{2P_2}^*)} = R_{2Q}$ holds for any distinct $P_1 \in \mathcal{P}ur(H_1)$ and $P_2 \in \mathcal{P}ur(H_1)$. Thus for all $Q \in \mathcal{P}ur(H_2)$, we have $\frac{M_{2P_1} Q M_{2P_1}^*}{\mathrm{Tr}(M_{2P_1} Q M_{2P_1}^*)} = \frac{M_{2P_2} Q M_{2P_2}^*}{\mathrm{Tr}(M_{2P_2} Q M_{2P_2}^*)}$, that is, $\frac{M_{2P} Q M_{2P}^*}{\mathrm{Tr}(M_{2P} Q M_{2P}^*)}$ is independent of P. So there exist $R_1 \in \mathcal{P}ur(K_1)$ and injective linear or conjugate linear operator $M_2 : H_2 \to K_2$ such that $\psi(P \otimes Q) = R_1 \otimes \frac{M_2 Q M_2^*}{\mathrm{Tr}(M_2 Q M_2^*)}$ for all separable pure states $P \otimes Q$.

Now, for any $B \in \mathcal{S}(H_2)$ and $P \in \mathcal{P}ur(H_1)$, writing $B = \sum_{j=1}^n b_j B_j$ as in the proof of Claim 4, we have $\psi(P \otimes B) = \psi(P \otimes \sum_{j=1}^n b_j B_j) = \sum_{j=1}^n b_j' \psi(P \otimes B_j) = \sum_{j=1}^n b_j' (R_1 \otimes \frac{M_2 B_j M_2^*}{\mathrm{Tr}(M_2 B_j M_2^*)}) = R_1 \otimes \sum_{j=1}^n b_j' \frac{M_2 B_j M_2^*}{\mathrm{Tr}(M_2 B_j M_2^*)}$ is a product state. But we already know that $\phi_2(P, \cdot) = \frac{M_2 (\cdot) M_2^*}{\mathrm{Tr}(M_2 (\cdot) M_2^*)}$, so $\mathrm{Tr}_1(\psi(P \otimes B)) = \phi_2(P, B) = \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}$. Thus $\psi(P \otimes B) = R_1 \otimes \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}$. Then, for any $A \in \mathcal{S}(H_1)$, writing $A = \sum_{i=1}^m a_i A_i$ as in the proof of Claim 4, we obtain $\psi(A \otimes B) = \psi(\sum_{i=1}^m a_i A_i \otimes B) = \sum_{i=1}^m a_i' \psi(A_i \otimes B) = \sum_{i=1}^m a_i' (R_1 \otimes \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}) = R_1 \otimes (\sum_{i=1}^m a_i' \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}))$, which is a product states. Therefore, we must have $\psi(A \otimes B) = R_1 \otimes \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}$, where M_2 is linear or conjugate linear. In the case that M_2 is a conjugate linear operator, it is well known that there exists a linear operator N_2 such that $\frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)} = \frac{N_2 B^t N_2^*}{\mathrm{Tr}(N_2 B^t N_2^*)}$ for all B. So the claim is true.

Similarly, one can show the following Claims 6-8.

Claim 6. If (a) and (c$'$) hold, then (4) holds, that is, there exist $R_2 \in \mathcal{P}ur(K_2)$ and injective linear or conjugate linear operator $M_1 : H_2 \to K_1$ such that

$$\psi(A \otimes B) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes R_2$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. In this case we must have $\dim H_2 \leq \dim K_1$.

Claim 7. If (b) and (a$'$) hold, then ψ has the form (5), that is, there exist $R_1 \in \mathcal{P}ur(K_1)$ and injective linear or conjugate linear operator $M_2 : H_1 \to K_2$ such that

$$\psi(A \otimes B) = R_1 \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. In this case we must have $\dim H_1 \leq \dim K_2$.

Claim 8. If (c) and (a′) hold, then there exist $R_2 \in \mathcal{P}ur(K_2)$ and injective linear or conjugate linear operator $M_1 : H_1 \to K_1$ such that

$$\psi(A \otimes B) = \frac{M_1 A M_1^*}{\mathrm{Tr}(M_1 A M_1^*)} \otimes R_2$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. In this case we must have $\dim H_1 \leq \dim K_1$. Hence ψ takes the form (2).

Claim 9. If (b) and (c′) hold, then there exist injective linear or conjugate linear (may not simultaneously) operators $M_1 : H_2 \to K_1$ and $M_2 : H_1 \to K_2$ such that

$$\psi(A \otimes B) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. In this case $\dim H_1 \leq \dim K_2$ and $\dim H_2 \leq \dim K_1$. So ψ has the form (7).

Suppose that (b) and (c′) hold; then $\phi_1(P,Q) = \phi_1(P_0,Q) = \frac{M_{1P_0} Q M_{1P_0}^*}{\mathrm{Tr}(M_{1P_0} Q M_{1P_0}^*)}$ and $\phi_2(P,Q) = \phi_2(P,Q_0) = \frac{M_{2Q_0} P M_{2Q_0}^*}{\mathrm{Tr}(M_{2Q_0} P M_{2Q_0}^*)}$. Thus, we obtain

$$\psi(P \otimes Q) = \phi_1(P,Q) \otimes \phi_2(P,Q) = \frac{M_{1P_0} Q M_{1P_0}^*}{\mathrm{Tr}(M_{1P_0} Q M_{1P_0}^*)} \otimes \frac{M_{2Q_0} P M_{2Q_0}^*}{\mathrm{Tr}(M_{2Q_0} P M_{2Q_0}^*)}$$

for all $P \in \mathcal{P}ur(H_1)$ and $Q \in \mathcal{P}ur(H_2)$, where $M_{1P_0} : H_2 \to K_1$ and $M_{2Q_0} : H_1 \to K_2$ are injective linear or conjugate linear operators. It follows that $\dim H_1 \leq \dim K_2$ and $\dim H_2 \leq \dim K_1$. Let $M_1 = M_{1P_0}$ and $M_2 = M_{2Q_0}$. Then $\psi(P \otimes Q) = \frac{M_1 Q M_1^*}{\mathrm{Tr}(M_1 Q M_1^*)} \otimes \frac{M_2 P M_2^*}{\mathrm{Tr}(M_2 P M_2^*)}$ for all separable pure states $P \otimes Q$. For any $A \otimes B \in \mathcal{S}(H_1 \otimes H_2)$, write $A = \sum_{i=1}^m a_i A_i$ and $B = \sum_{j=1}^n b_j B_j$ by the spectral theorem, where $a_i, b_j \geq 0$ with $\sum_{i=1}^m a_i = 1, \sum_{j=1}^n b_j = 1$ and A_is, B_js pure states. Thus, for any $Q \in \mathcal{P}ur(H_2)$, $\psi(A \otimes Q) = \psi(\sum_{i=1}^m a_i A_i \otimes Q) = \sum_{i=1}^m a_i' \psi(A_i \otimes Q) = \sum_{i=1}^m a_i' (\frac{M_1 Q M_1^*}{\mathrm{Tr}(M_1 Q M_1^*)} \otimes \frac{M_2 A_i M_2^*}{\mathrm{Tr}(M_2 A_i M_2^*)}) = \frac{M_1 Q M_1^*}{\mathrm{Tr}(M_1 Q M_1^*)} \otimes (\sum_{i=1}^m a_i' \frac{M_2 A_i M_2^*}{\mathrm{Tr}(M_2 A_i M_2^*)})$ is a product state. As $\phi_2(\cdot, Q) = \frac{M_2 (\cdot) M_2^*}{\mathrm{Tr}(M_2 (\cdot) M_2^*)}$, we obtain that $\mathrm{Tr}_1(\psi(A \otimes Q)) = \phi_2(A,Q) = \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}$. So we must have $\psi(A \otimes Q) = \frac{M_1 Q M_1^*}{\mathrm{Tr}(M_1 Q M_1^*)} \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}$. It follows that $\psi(A \otimes B) = \psi(A \otimes \sum_{j=1}^n b_j B_j) = \sum_{j=1}^n b_j' \psi(A \otimes B_j) = \sum_{j=1}^n b_j' (\frac{M_1 B_j M_1^*}{\mathrm{Tr}(M_1 B_j M_1^*)} \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}) = (\sum_{j=1}^n b_j' \frac{M_1 B_j M_1^*}{\mathrm{Tr}(M_1 B_j M_1^*)}) \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}$. On the other hand, we also have $\psi(P \otimes B) = \psi(P \otimes \sum_{j=1}^n b_j B_j) = \sum_{j=1}^n b_j' \psi(P \otimes B_j) = \sum_{j=1}^n b_j' (\frac{M_1 B_j M_1^*}{\mathrm{Tr}(M_1 B_j M_1^*)} \otimes \frac{M_2 P M_2^*}{\mathrm{Tr}(M_2 P M_2^*)}) = (\sum_{j=1}^n b_j' \frac{M_1 B_j M_1^*}{\mathrm{Tr}(M_1 B_j M_1^*)}) \otimes \frac{M_2 P M_2^*}{\mathrm{Tr}(M_2 P M_2^*)}$. As $\phi_1(P, \cdot) = \frac{M_1 (\cdot) M_1^*}{\mathrm{Tr}(M_1 (\cdot) M_1^*)}$, then $\mathrm{Tr}_2(\psi(P \otimes B)) = \phi_1(P,B) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)}$. Thus we get $\psi(P \otimes B) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes \frac{M_2 P M_2^*}{\mathrm{Tr}(M_2 P M_2^*)}$ and then $\psi(A \otimes B) = \psi(\sum_{i=1}^m a_i A_i \otimes B) = \sum_{i=1}^m a_i \psi(A_i \otimes B) = \sum_{i=1}^m a_i' (\frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes \frac{M_2 A_i M_2^*}{\mathrm{Tr}(M_2 A_i M_2^*)}) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes (\sum_{i=1}^m a_i' \frac{M_2 A_i M_2^*}{\mathrm{Tr}(M_2 A_i M_2^*)})$. Now it is clear that $\psi(A \otimes B) = \frac{M_1 B M_1^*}{\mathrm{Tr}(M_1 B M_1^*)} \otimes \frac{M_2 A M_2^*}{\mathrm{Tr}(M_2 A M_2^*)}$ for any $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. Hence the claim is true.

Similarly, we have

Claim 10. If (c) and (b′) hold, then there exist injective linear or conjugate linear (may not simultaneously) operators $M_1 : H_1 \to K_1$ and $M_2 : H_2 \to K_2$ such that

$$\psi(A \otimes B) = \frac{M_1 A M_1^*}{\mathrm{Tr}(M_1 A M_1^*)} \otimes \frac{M_2 B M_2^*}{\mathrm{Tr}(M_2 B M_2^*)}$$

for all $A \in \mathcal{S}(H_1)$ and $B \in \mathcal{S}(H_2)$. In this case $\dim H_i \leq K_i$, $i = 1, 2$. Hence in this case ψ has the form (6).

Claim 11. If (b) and (b′) hold, then ψ has the form (9).

Assume (b) and (b′) hold synchronously. Then for any $P \otimes Q \in \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$ we have $\psi(P \otimes Q) = \phi_1(P, Q) \otimes \phi_2(P, Q) = R_{1Q} \otimes \frac{M_{2Q} P M_{2Q}^*}{\mathrm{Tr}(M_{2Q} P M_{2Q}^*)} = R_{1P} \otimes \frac{M_{2P} Q M_{2P}^*}{\mathrm{Tr}(M_{2P} Q M_{2P}^*)}$. It follows that there exist $R_1 \in \mathcal{P}ur(K_1)$ such that $R_{1Q} = R_{1P} = R_1$ and $\frac{M_{2Q} P M_{2Q}^*}{\mathrm{Tr}(M_{2Q} P M_{2Q}^*)} = \frac{M_{2P} Q M_{2P}^*}{\mathrm{Tr}(M_{2P} Q M_{2P}^*)}$ for all P, Q. Thus there exists a strict convex combination preserving map $\varphi_2 : \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2) \to \mathcal{S}(K_2)$ such that, for each $P \otimes Q \in \mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)$, $\varphi_2(P \otimes Q) = \frac{M_P Q M_P^*}{\mathrm{Tr}(M_P Q M_P^*)} = \frac{N_Q P N_Q^*}{\mathrm{Tr}(N_Q P N_Q^*)}$ for some injective, may not synchronously, linear or conjugate linear operators $M_P : H_2 \to K_2$, $N_Q : H_1 \to K_2$, and

$$\psi(\rho) = R_1 \otimes \varphi_2(\rho)$$

for all $\rho \in \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2)$. In this case $\max\{\dim H_1, \dim H_2\} \leq \dim K_2$. So ψ has the form (9) and Claim 11 is true.

Similarly,

Claim 12. If (c) and (c′) hold, then ψ takes the form (8).

Combining the claims 4-12, we complete the proof of Theorem 3.2. □

By Theorem 3.2, the following corollary is immediate, which gives an affirmative answer to a conjecture in [8], that is, the conjecture (4°), without the injectivity assumption.

Corollary 3.3 *Let* $\psi : \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2) \to \mathcal{S}_{\mathrm{sep}}(K_1 \otimes K_2)$ *be a map with* $2 \leq \dim H_i < \infty$, $i = 1, 2$, *and* $\mathrm{ran}\psi$ *non-collinear or a singleton (i.e., contains only one element). If* ψ *preserves separable pure states and strict convex combinations, then it sends product states to product states.*

Now we give a characterization of injective local quantum measurements, which reveals that, in almost all situations the maps preserving separable pure states and strict convex combinations are essentially the injective local quantum measurements.

Corollary 3.4 *Let* H_1, H_2, K_1, K_2 *be Hilbert spaces with* $2 \leq \dim H_i < \infty$, $i = 1, 2$ *and let* $\psi : \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2) \to \mathcal{S}_{\mathrm{sep}}(K_1 \otimes K_2)$ *be a map. Then the following statements are equivalent.*

(1) ψ *is strict convex combination preserving,* $\psi(\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2)) \subseteq \mathcal{P}ur(K_1) \otimes \mathcal{P}ur(K_2)$ *and the range of* ψ *is non-collinear containing a state* σ *so that both reductions* $\mathrm{Tr}_1(\sigma)$ *and* $\mathrm{Tr}_2(\sigma)$ *have rank* ≥ 2.

(2) *ψ is open line segment preserving, $\psi(\mathcal{P}ur(H_1)\otimes\mathcal{P}ur(H_2))\subseteq\mathcal{P}ur(K_1)\otimes\mathcal{P}ur(K_2)$ and the range of ψ is non-collinear containing a state σ so that both reductions $\mathrm{Tr}_1(\sigma)$ and $\mathrm{Tr}_2(\sigma)$ have rank ≥ 2.*

(3) *Either*

(1°) *there exist injective operators $M_1\in\mathcal{B}(H_1,K_1)$ and $M_2\in\mathcal{B}(H_2,K_2)$ such that*

$$\psi(\rho)=\frac{(M_1\otimes M_2)\Phi(\rho)(M_1\otimes M_2)^*}{\mathrm{Tr}((M_1\otimes M_2)\Phi(\rho)(M_1\otimes M_2)^*)}$$

for all $\rho\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$;

or

(2°) *there exist injective operators $M_1\in\mathcal{B}(H_2,K_1)$ and $M_2\in\mathcal{B}(H_1,K_2)$ such that*

$$\psi(\rho)=\frac{(M_1\otimes M_2)\Phi(\Theta(\rho))(M_1\otimes M_2)^*}{\mathrm{Tr}((M_1\otimes M_2)\Phi(\Theta(\rho))(M_1\otimes M_2)^*)}$$

for all $\rho\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$.

Here Φ is the identity, or the transpose, or the partial transpose of the first system or the partial transpose of the second system with respect to an arbitrarily fixed product basis, Θ is the swap.

Proof. (1)$\Leftrightarrow$(2)$\Leftarrow$(3) is obvious, we need to check the (1)$\Rightarrow$(3).

Assume (1). It is clear that ψ has one of the forms Theorem 3.2.(1)-(10) as ψ satisfies all the conditions of Theorem 3.2. Furthermore, the assumption that there exists σ in ranψ so that rank$\mathrm{Tr}_i(\sigma)\geq 2$, $i=1,2$ forces that ψ can only take the form (6) or (7), that is, $\psi(A\otimes B)=\frac{M_1\Psi_1(A)M_1^*}{\mathrm{Tr}(M_1\Psi_1(A)M_1^*)}\otimes\frac{M_2\Psi_2(B)M_2^*}{\mathrm{Tr}(M_2\Psi_2(B)M_2^*)}$ for all $A\otimes B\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$, or $\psi(A\otimes B)=\frac{M_1\Psi_2(B)M_1^*}{\mathrm{Tr}(M_1\Psi_2(B)M_1^*)}\otimes\frac{M_2\Psi_1(A)M_2^*}{\mathrm{Tr}(M_2\Psi_1(A)M_2^*)}$ for all $A\otimes B\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$, where Ψ_i is the identity or the transpose with respect to an arbitrarily fixed orthonormal basis, $i=1,2$. Thus either

(i) $\psi(A\otimes B)=\frac{(M_1\otimes M_2)\Phi(A\otimes B)(M_1\otimes M_2)^*}{\mathrm{Tr}(M_1\otimes M_2)\Phi(A\otimes B)(M_1\otimes M_2)^*)}$ for all $A\otimes B\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$; or

(ii) $\psi(A\otimes B)=\frac{(M_1\otimes M_2)\Phi(\Theta(A\otimes B))(M_1\otimes M_2)^*}{\mathrm{Tr}(M_1\otimes M_2)\Phi(\Theta(A\otimes B))(M_1\otimes M_2)^*)}$ for all $A\otimes B\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$,

where Φ is the identity, or the transpose, or the partial transpose of the first system or the partial transpose of the second system with respect to an arbitrarily fixed product basis, Θ is the swap.

Let

$$\Delta(\rho)=\frac{(M_1^{[-1]}\otimes M_2^{[-1]})\psi(\Phi(\rho))(M_1^{[-1]}\otimes M_2^{[-1]})^*}{\mathrm{Tr}((M_1^{[-1]}\otimes M_2^{[-1]})\psi(\Phi(\rho))(M_1^{[-1]}\otimes M_2^{[-1]})^*)}$$

if ψ has form (i), and let

$$\Delta(\rho)=\frac{\Theta[(M_1^{[-1]}\otimes M_2^{[-1]})\psi(\Phi(\rho))(M_1^{[-1]}\otimes M_2^{[-1]})^*]}{\mathrm{Tr}((M_1^{[-1]}\otimes M_2^{[-1]})\psi(\Phi(\rho))(M_1^{[-1]}\otimes M_2^{[-1]})^*)}$$

if ψ takes the form (ii). Then $\Delta : \mathcal{S}_{\rm sep}(H_1\otimes H_2)\to\mathcal{S}_{\rm sep}(H_1\otimes H_2)$ is a bijective map preserving separable pure states and strict convex combinations. Furthermore $\Delta(A\otimes B)=A\otimes B$ for all $A\otimes B\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$. Then by [8, Lemma 2.7], we get $\Delta(\rho)=\rho$ for any $\rho\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$. Now it is clear that ψ has the form (1°) or (2°), finishing the proof. $\Box$

The following result is a generalization of the main result in [8] by omitting the additional assumption that "for any $P_i\in\mathcal{S}(H_i)$ with rank$P_i=1$ and rank$P_j=2$ $(1\le i\ne j\le 2)$, there exist $P_i'\in\mathcal{S}(H_i)$ $(i=1,2)$ such that $\Phi(P_1\otimes P_2)=P_1'\otimes P_2'$" for finite dimensional case, and thus, answer affirmatively a conjecture proposed in [8], i.e., the conjecture (3°), as promised in the introduction section.

Theorem 3.5 *Let H_i, K_i, be complex Hilbert spaces with $2\le\dim H_i<\infty$, $i=1,2$, and $\psi:\mathcal{S}_{\rm sep}(H_1\otimes H_2)\to\mathcal{S}_{\rm sep}(K_1\otimes K_2)$ an bijective map. Then Φ is convex combination preserving if and only if either*

(1) *there exist invertible operators $S\in\mathcal{B}(H_1,K_1)$ and $T\in\mathcal{B}(H_2,K_2)$ such that*

$$\psi(\rho)=\frac{(S\otimes T)\Psi(\rho)(S\otimes T)^*}{{\rm Tr}((S\otimes T)\Psi(\rho)(S\otimes T)^*)}$$

for all $\rho\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$;

or

(2) *there exist invertible operators $S\in\mathcal{B}(H_2,K_1)$ and $T\in\mathcal{B}(H_1,K_2)$ such that*

$$\psi(\rho)=\frac{(S\otimes T)\Psi(\Theta(\rho))(S\otimes T)^*}{{\rm Tr}((S\otimes T)\Psi(\Theta(\rho))(S\otimes T)^*)}$$

for all $\rho\in\mathcal{S}_{\rm sep}(H_1\otimes H_2)$.

Here Ψ is the identity, or the transpose, or the partial transpose of the first system or the partial transpose of the second system with respect to an arbitrarily fixed product basis, Θ is the swap.

Proof. We need only check the "only if" part. By the assumption, ψ is bijective and strict convex combination preserving from $\mathcal{S}_{\rm sep}(H_1\otimes H_2)$ onto $\mathcal{S}_{\rm sep}(K_1\otimes K_2)$. Particularly, the range of ψ is non-collinear. By [8], we have $\psi(\mathcal{P}ur_{\rm sep}(H_1\otimes H_2))=\mathcal{P}ur_{\rm sep}(K_1\otimes K_2)$, that is, ψ preserves separable pure states in both directions.

Then by Corollary 3.4, ψ has either the form (1) or the form (2) with T,S injective. Since $\psi(\mathcal{P}ur_{\rm sep}(H_1\otimes H_2))=\mathcal{P}ur_{\rm sep}(K_1\otimes K_2)$, it is clear that both T and S are invertible. Hence the theorem is true. $\Box$

4. Maps preserving separable pure states and strict convex combinations: Multipartite systems

The results similar to that in Section 3 for bipartite case are valid for multipartite cases, of course, with more complicated expressions. The proofs are also similar. In this section we only list some of them, which have relatively simple expressions and may have more applications. The meanings of the notations used here are also similar to that in Section 3.

Suppose $\dim H_i = n_i$. For $1 \leq r_1 < \cdots < r_p \leq n$, define the partial trace which is a linear map $\mathrm{Tr}^{r_1,\cdots,r_p} : \mathcal{B}_{\mathrm{sa}}(\otimes_{i=1}^n H_i) \to \mathcal{B}_{\mathrm{sa}}(\otimes_{j=1}^p H_{r_j})$ as follows:

$$\otimes_{i=1}^n A_i \longmapsto (\prod_{i\neq r_1,\cdots,r_p} \mathrm{Tr} A_i) \otimes_{j=1}^p A_{r_j}.$$

In particular, the linear map $\mathrm{Tr}^r : \mathcal{B}_{\mathrm{sa}}(\otimes_{i=1}^n H_i) \to \mathcal{B}_{\mathrm{sa}}(H_r)$ is given by $\mathrm{Tr}^r(\otimes_{i=1}^n A_i) = (\prod_{i\neq r} \mathrm{Tr}(A_i))A_r$. We call $\mathrm{Tr}^r(\rho)$ the reduction state of $\rho \in \mathcal{S}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$ in the subsystem $\mathcal{S}(H_r)$. $\rho \in \mathcal{S}(\otimes_{i=1}^n H_i)$ is called a product state if $\rho = \otimes_{i=1}^n \rho_i$ for some $\rho_i \in \mathcal{S}(H_i)$.

The following result corresponds to Corollary 3.3 and Corollary 3.4.

Theorem 4.1 *Let $\psi : \mathcal{S}_{\mathrm{sep}}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{S}_{\mathrm{sep}}(K_1 \otimes K_2 \otimes \cdots \otimes K_n)$ be a strict convex combination preserving map, with $2 \leq \dim H_i < \infty$, $i = 1, 2, \cdots, n$, and $\psi(\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2) \otimes \cdots \otimes \mathcal{P}ur(H_n)) \subseteq \mathcal{P}ur(K_1) \otimes \mathcal{P}ur(K_2) \otimes \cdots \otimes \mathcal{P}ur(K_n)$.*

(1) *If the range of ψ is non-collinear or a singleton, then ψ maps product states to product states.*

(2) *If the range of ψ is non-collinear and contains a state σ so that its reduction state $\mathrm{Tr}^i(\sigma)$ has rank ≥ 2 for each $i = 1, 2, \ldots, n$, then there exist a permutation $\pi : (1, \cdots, n) \mapsto (\pi(1), \cdots, \pi(n))$ of $(1, \cdots, n)$ and injective linear or conjugate linear (may not simultaneously) operators $M_j : H_{\pi(j)} \to K_j$, $j = 1, \cdots, n$, such that*

$$\psi(\rho) = \frac{(M_1 \otimes \cdots \otimes M_n)\Theta_\pi(\rho)(M_1^* \otimes \cdots \otimes M_n^*)}{\mathrm{Tr}((M_1 \otimes \cdots \otimes M_n)\Theta_\pi(\rho)(M_1^* \otimes \cdots \otimes M_n^*))}$$

for all $\rho \in \mathcal{S}_{\mathrm{sep}}(H_1 \otimes \cdots \otimes H_n)$. Here $\Theta_\pi : \mathcal{B}_{\mathrm{sa}}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{B}_{\mathrm{sa}}(H_{\pi(1)} \otimes H_{\pi(2)} \otimes \cdots \otimes H_{\pi(n)})$ is a linear map determined by $\Theta_\pi(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)}$. It is clear that $\dim H_{\pi(j)} \leq \dim K_j$.

Proof. We give a skeleton of the proof. If the range of ψ is collinear, it is clear that either ψ contracts to a pure state or there exist distinct separable pure state σ_1 and σ_2 such that $\psi(\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2) \otimes \cdots \otimes \mathcal{P}ur(H_n)) = \{\sigma_1, \sigma_2\}$ and $\mathrm{ran}(\psi) \subseteq [\sigma_1, \sigma_2]$. Thus, in the sequel, we assume the range of ψ is non-collinear. For $r = 1, \cdots, n$, define maps $\phi_r : (\mathcal{S}(H_1), \cdots, \mathcal{S}(H_n)) \to \mathcal{S}(K_r)$ by

$$\phi_r(A_1, \cdots, A_n) = \mathrm{Tr}^r(\psi(\otimes_{i=1}^n A_i)) \ for\ all\ (A_1, \cdots, A_n) \in (\mathcal{S}(H_1), \cdots, \mathcal{S}(H_n)).$$

Notice that

$$\psi(\otimes_{i=1}^n P_i) = \otimes_{r=1}^n \phi_r(P_1, \cdots, P_n) \; for \; all \; (P_1, \cdots, P_n) \in (\mathcal{P}ur(H_1), \cdots, \mathcal{P}ur(H_n)).$$

Given arbitrary $Q_i \in \mathcal{P}ur(H_i)$ for $i = 2, \cdots, n$, the map $\phi_r(\cdot, Q_2, \cdots, Q_n)$ maps $\mathcal{P}ur(H_1)$ into $\mathcal{P}ur(K_r)$ and also preserves strict convex combination. Then by Theorem 2.5, the map $\phi_r(\cdot, Q_2, \cdots, Q_n)$ must have the form (1) or (2) or (3) stated in Theorem 2.5.

Firstly we claim that

Claim 1. $\phi_r(\cdot, Q_2, \cdots, Q_n)$, $r = 1, \cdots, n$, can not have the form Theorem 2.5.(2) for any choice $(Q_2, \ldots, Q_n)$ with $Q_i \in \mathcal{P}ur(H_i)$.

As ψ is strict convex combination preserving and ranψ is non-collinear, by Lemma 2.3, we have $\psi(\rho) = \frac{\Gamma(\rho)+B}{f(\rho)+b}$ for all $\rho \in \mathcal{S}_{\text{sep}}(H_1 \otimes \cdots \otimes H_n)$, where $\Gamma : \mathcal{B}_{\text{sa}}(H_1 \otimes \cdots \otimes H_n) \to \mathcal{B}_{\text{sa}}(H_1 \otimes \cdots \otimes H_n)$ is a linear map, $B \in \mathcal{B}_{\text{sa}}(H_1 \otimes \cdots \otimes H_n)$ is an operator, $f : \mathcal{B}_{\text{sa}}(H_1 \otimes \cdots \otimes H_n) \to \mathbb{R}$ is a linear functional and $b \in \mathbb{R}$ with $f(\rho) + b > 0$ for any $\rho \in \mathcal{S}_{\text{sep}}(H_1 \otimes \cdots \otimes H_n)$. Since $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ is finite dimensional, Γ, f are continuous. It follows that ψ is continuous and so $\phi_r(\cdot, Q_2, \cdots, Q_n)$ is continuous on $\mathcal{S}(H_1)$ for each $r = 1, \cdots, n$, this ensures that $\phi_r(\cdot, Q_2, \cdots, Q_n)$ can not be the form Theorem 2.5.(2).

Claim 2. For any $r \in \{1, 2, \ldots, n\}$, either $\phi_r(\cdot, Q_2, \cdots, Q_n)$ has the form Theorem 2.5.(1) for all $Q_i \in \mathcal{P}ur(H_i)$ or $\phi_r(\cdot, Q_2, \cdots, Q_n)$ has the form Theorem 2.5.(3) for all $Q_i \in \mathcal{P}ur(H_i)$.

Fix $A_1 \in \mathcal{S}(H_1)$ with rank$A_1 \geq 2$. Define $F_{A_1,r} : (\mathcal{P}ur(H_2), \cdots, \mathcal{P}ur(H_n)) \to \mathbb{R}$ by

$$F_{A_1,r}(Q_2, \cdots, Q_n) = \|\phi_r(A_1, Q_2, \cdots, Q_n)\|_2.$$

It is clear that, when $\phi_r(\cdot, Q_2, \cdots, Q_n)$ has the form Theorem 2.5.(1), $F_{A_1,r}(Q_2, \cdots, Q_n) = 1$; when $\phi_r(\cdot, Q_2, \cdots, Q_n)$ has the form Theorem 2.5.(3), $F_{A_1,r}(Q_2, \cdots, Q_n) < 1$. Then similar to the argument of Claim 1 in the proof of Theorem 3.2, we obtain that either $\phi_r(\cdot, Q_2, \cdots, Q_n)$ has the form Theorem 2.5.(1) for all $Q_i \in \mathcal{P}ur(H_i)$ or $\phi_r(\cdot, Q_2, \cdots, Q_n)$ has the form Theorem 2.5.(3) for all $Q_i \in \mathcal{P}ur(H_i)$.

Claim 3. For any choice $(Q_2, \ldots, Q_n) \in (\mathcal{P}ur(H_2), \ldots, \mathcal{P}ur(H_n))$, at most one of the maps $\phi_r(\cdot, Q_2, \cdots, Q_n)$, $r = 1, \cdots, n$, has the form Theorem 2.5.(3) and all the exceptional maps have the form Theorem 2.5.(1).

Assume that there exists $(Q_2, \ldots, Q_n)$ so that both the maps

$$\phi_s(\cdot, Q_2, \cdots, Q_n) \text{ and } \phi_t(\cdot, Q_2, \cdots, Q_n)$$

have the form Theorem 2.5.(3) for some distinct s, t. Then consider the map $L : \mathcal{S}(H_1) \to \mathcal{S}(K_s \otimes K_t)$ defined by $L(A) = \text{Tr}^{s,t}(\psi(A \otimes (\otimes_{i=2}^n Q_i)))$. Obviously, L is strict convex combination preserving and $L(P) = \phi_s(P, Q_2, \cdots, Q_n) \otimes \phi_t(P, Q_2, \cdots, Q_n) \in \mathcal{P}ur(K_s \otimes K_t)$ for

all $P \in \mathcal{P}ur(H_1)$. Because $\phi_s(\cdot, Q_2, \cdots, Q_n)$ and $\phi_t(\cdot, Q_2, \cdots, Q_n)$ are of the form Theorem 2.5.(3), there exist injective linear or conjugate linear operators M_s and M_t such that $\phi_s(P, Q_2, \cdots, Q_n) = \frac{M_s P M_s^*}{\mathrm{Tr}(M_s P M_s^*)}$ and $\phi_t(P, Q_2, \cdots, Q_n) = \frac{M_t P M_t^*}{\mathrm{Tr}(M_t P M_t^*)}$, respectively. It follows that

$$L(P) = \frac{(M_s \otimes M_t)(P \otimes P)(M_s \otimes M_t)^*}{\mathrm{Tr}((M_s \otimes M_t)(P \otimes P)(M_s \otimes M_t)^*)}.$$

Following the same argument as Claim 2 in the proof of Theorem 3.2, one can get a contradiction.

Applying the same argument on the map $\phi_r(Q_1, \cdots, Q_{p-1}, (\cdot), Q_{p+1}, \cdots, Q_n) : \mathcal{S}(H_p) \to \mathcal{S}(K_r)$, one sees that Claim 1 to Claim 3 also hold for any $p = 2, \cdots, n$.

Now, by a similar approach as that in the proof of Theorem 3.2, it is easily seen that one of the following holds: (i) ψ contracts to a separable pure state; (ii) $\psi(\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2) \otimes \cdots \mathcal{P}ur(H_n))$ contains two separable pure states $\sigma 1, \sigma 2$ and $\mathrm{ran}(\psi) \subseteq [\sigma_1, \sigma_2]$; (iii) $\mathrm{ran}(\psi)$ is non-collinear, ψ maps product states to product states and, for any $\rho \in \mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$, at least one of reductions $\mathrm{Tr}^i(\psi(\rho))$, $i = 1, 2, \ldots, n$, is a pure state; (iv) ψ sends product states to product states and there is a permutation $(\pi(1), \cdots, \pi(n))$ of $(1, \cdots, n)$ such that $\phi_{\pi^{-1}(p)}(Q_1, \cdots, Q_{p-1}, (\cdot), Q_{p+1}, \cdots, Q_n)$ has the form Theorem 2.5.(3) for all $Q_i \in \mathcal{P}ur(H_i)$.

Therefore, if the range of ψ is non-collinear or singleton, then ψ sends product states to product states, that is, the statement (1) holds.

Furthermore, if there exists $\rho \in \mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$ such that all reductions $\mathrm{Tr}^i(\psi(\rho))$, $i = 1, 2, \ldots, n$, have rank ≥ 2, then only (iv) can hold. Thus, there exist a permutation π of $(1, 2, \ldots, n)$ and injective linear or conjugate linear (may not simultaneously) operators $M_i : H_{\pi(i)} \to K_i$ such that $\phi_i(Q_1, \cdots, P_{\pi(i)}, \cdots, Q_n) : \mathcal{S}(H_{\pi(i)}) \to \mathcal{S}(K_i)$ is of the form $\rho_{\pi(i)} \mapsto \frac{M_i \rho_{\pi(i)} M_i^*}{\mathrm{Tr}(M_i \rho_{\pi(i)} M_i^*)}$. Moreover,

$$\begin{aligned} & \psi(P_1 \otimes \cdots \otimes P_n) = \phi_1(P_1, \cdots, P_n) \otimes \cdots \otimes \phi_n(P_1, \cdots, P_n) \\ = \ & \phi_1(Q_1, \cdots, P_{\pi(1)}, \cdots, Q_n) \otimes \cdots \otimes \phi_n(Q_1, \cdots, P_{\pi(n)}, \cdots, Q_n) \\ = \ & \tfrac{M_1 P_{\pi(1)} M_1^*}{\mathrm{Tr}(M_1 P_{\pi(1)} M_1^*)} \otimes \cdots \otimes \tfrac{M_n P_{\pi(n)} M_n^*}{\mathrm{Tr}(M_n P_{\pi(n)} M_n^*)} \end{aligned}$$

for any $P_1 \otimes \cdots \otimes P_n \in \mathcal{P}ur(H_1) \otimes \cdots \otimes \mathcal{P}ur(H_n)$. Then it is easily checked that

$$\psi(A_1 \otimes \cdots A_n) = \frac{M_1 A_{\pi(1)} M_1^*}{\mathrm{Tr}(M_1 A_{\pi(1)} M_1^*)} \otimes \cdots \otimes \frac{M_n A_{\pi(n)} M_n^*}{\mathrm{Tr}(M_n A_{\pi(n)} M_n^*)}$$

for any $A_1 \otimes \cdots \otimes A_n \in \mathcal{S}_{\rm sep}(H_1 \otimes \cdots \otimes H_n)$. Now an argument similar to the proof of Corollary 3.4 entails that

$$\psi(\rho) = \frac{(M_1 \otimes \cdots \otimes M_n)\Theta_\pi(\rho)(M_1 \otimes \cdots \otimes M_n)^*}{\mathrm{Tr}((M_1 \otimes \cdots \otimes M_n)\Theta_\pi(\rho)(M_1 \otimes \cdots \otimes M_n)^*)}$$

holds for all $\rho \in \mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$. Here $\Theta_\pi : \mathcal{B}_{\rm sa}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{B}_{\rm sa}(H_{\pi(1)} \otimes H_{\pi(2)} \otimes \cdots \otimes H_{\pi(n)})$ is the linear map determined by $\Theta_\pi(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)}$. Obviously $\dim H_{\pi(i)} \leq \dim K_i$, $i = 1, \cdots, n$. So the statement (2) is true. $\square$

The following result is corresponding to Theorem 3.5.

Corollary 4.2 *Let $\psi : \mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{S}_{\rm sep}(K_1 \otimes K_2 \otimes \cdots \otimes K_n)$ be a bijective map with $2 \leq \dim H_i < \infty$, $i = 1, 2, \cdots, n$. Then ψ is strict convex combination preserving if and only if there exist a permutation π of $(1, 2, \cdots, n)$ and invertible linear or conjugate linear (may not simultaneously) operators $M_j : H_{\pi(j)} \to K_j$, $j = 1, \cdots, n$, such that*

$$\psi(\rho) = \frac{(M_1 \otimes \cdots \otimes M_n)\Theta_\pi(\rho)(M_1^* \otimes \cdots \otimes M_n^*)}{\mathrm{Tr}((M_1 \otimes \cdots \otimes M_n)\Theta_\pi(\rho)(M_1^* \otimes \cdots \otimes M_n^*))} \tag{4.1}$$

holds for all $\rho \in \mathcal{S}_{\rm sep}(H_1 \otimes \cdots \otimes H_n)$. Here $\Theta_\pi : \mathcal{B}_{\rm sa}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{B}_{\rm sa}(H_{\pi(1)} \otimes H_{\pi(2)} \otimes \cdots \otimes H_{\pi(n)})$ is the linear map determined by $\Theta_\pi(A_1 \otimes A_2 \otimes \cdots \otimes A_n) = A_{\pi(1)} \otimes A_{\pi(2)} \otimes \cdots \otimes A_{\pi(n)}$. It is clear that $\dim H_{\pi(j)} = \dim K_j$.

Proof. For "only if" part, it is clear that the assumptions on ψ implies that $\psi(\mathcal{P}ur(H_1) \otimes \mathcal{P}ur(H_2) \otimes \cdots \otimes \mathcal{P}ur(H_n)) = \mathcal{P}ur(K_1) \otimes \mathcal{P}ur(K_2) \otimes \cdots \otimes \mathcal{P}ur(K_n)$. Then by Theorem 4.1, ψ has the form Eq.(4.1) with each M_j bijective. The "if" part is obvious. $\square$

5. Conclusion

Let $M \in \mathcal{B}(H, K)$. The quantum measurement map ϕ_M defined by $\phi_M(\rho) = \frac{M\rho M^*}{\mathrm{Tr}(M\rho M^*)}$ from the convex subset $\mathcal{S}_M(H) = \{\rho : M\rho M^* \neq 0\}$ of the (convex) set $\mathcal{S}(H)$ of states into $\mathcal{S}(K)$ is strict convex combination preserving and sends pure states in $\mathcal{S}_M(H)$ to pure states. Similarly, each local quantum measurement map $\rho \mapsto \frac{(M_1 \otimes M_2)\rho(M_1^* \otimes M_2^*)}{\mathrm{Tr}((M_1 \otimes M_2)\rho(M_1^* \otimes M_2^*))}$ preserves separable pure states and strict convex combinations. Above facts make it interesting to study the problem of characterizing the maps between convex subsets of states in (multipartite) quantum systems that are (separable) pure state preserving and strict convex combination preserving, that is, the problems (1°) and (2°) mentioned in the introduction section. These problems are basic and interesting both in quantum information science and mathematics science, and their solutions will present a geometric characterization of (local) quantum measurements and help us to understand better the quantum measurement.

Though we are not able to solve these problems thoroughly in the present paper, we can give a characterization of the maps $\psi : \mathcal{S}(H) \to \mathcal{S}(K)$ with $2 \leq \dim H < \infty$ that preserve pure states and strict convex combinations, and give a structure theorem of the maps $\phi : \mathcal{S}(H_1 \otimes H_2 \otimes \cdots \otimes H_n) \to \mathcal{S}(K_1 \otimes K_2 \otimes \cdots \otimes K_n)$ with $2 \leq \dim H_i < \infty$ that preserve separable pure states and strict convex combinations. From these results we get a characterization of injective (local) quantum measurements. In almost all situations, for example, if the range of

ϕ is non-collinear or a singleton, then ϕ sends product states to product states. In particular, if the range of ϕ is non-collinear and contains an element with each reduction having rank ≥ 2, then ϕ is essentially an injective local quantum measurement. Thus we answer affirmatively two conjectures proposed in [8].

It is now clear that, to solve the above problems (1°) and (2°), one have to consider the situations that the domains of the maps are not the whole set $\mathcal{S}(H)$ ($\mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$). We may assume that the domain of the strict convex combination preserving and (separable) pure state preserving maps are those convex subset which are complement of some face of $\mathcal{S}(H)$ ($\mathcal{S}_{\rm sep}(H_1 \otimes H_2 \otimes \cdots \otimes H_n)$).

Finally we remark that, the main results obtained in [5, 8] hold for both finite dimensional systems and infinite dimensional systems. However, in the present paper we only deal with finite dimensional systems. We conjecture that the results in this paper are also valid for infinite dimensional case.

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